# Fundamental concepts in representation theory
(notes for course taught at HUJI, Spring 2020)
(UNPOLISHED DRAFT)

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## Contents

# 1 Introduction, conventions, notations, etc.

These notes are very preliminary, contain a lot of unfinished things, etc.! I don't claim originality etc.!

# 2 Basic representation theory

In this section, we discuss the basics of representation theory of finite groups.

Throughout this section, we fix a group $G$ and a ground field $k$ (so that all vector spaces are $k$-vector spaces, etc.).

## 2.1 $G$-sets and $G$-representations

### 2.1.1

Recall that groups originated as the collections of **symmetries** of objects. At its barest, an object is a set $X$ with some extra structure and a symmetry is a bijection $X \to X$ which preserves the extra structure. Then we expect that the identity $id_X : X \to X$ is a symmetry, if $\sigma, \tau : X \to X$ are symmetries then so is $\sigma \circ \tau$, and if $\sigma : X \to X$ is a symmetry then so is $\sigma^{-1}$. In other words, we expect the set of symmetries to form a subgroup of the group of all bijections $X \to X$ under composition.

**Example 2.1.** *We can consider the empty extra structure on the set* $\{1, \dots, n\}$*. Then the group of symmetries is* $S_n$*, the group of bijections* $\{1, \dots, n\} \to \{1, \dots, n\}$ *under composition. More generally, for a set* $X$*, we will denote by* $S_X$ *the group of bijections* $X \to X$ *under composition.*

**Example 2.2.** *We can consider the extra structure on* $\{1, \dots, n\}$ *to be the subset* $\{1, \dots, m\}$*. So symmetries are the bijections* $\{1, \dots, n\} \to \{1, \dots, n\}$ *which preserve the subset* $\{1, \dots, m\}$*. We obtain a subgroup of* $S_n$ *isomorphic to* $S_m \times S_{n-m}$*.*

**Example 2.3.** *Let* $V$ *be a* $k$*-vector space. The extra structure on* $V$ *of a linear vector space specifies a subgroup* $GL(V) \subset S_V$*, consisting of the linear automorphisms of* $V$*.*

### 2.1.2

So, groups are an abstraction of collections of symmetries. A reversal of this is then, for our abstract group $G$, to search for objects of which it can be thought of as a collection of symmetries.

**Definition 2.4.** A **$G$-set** is a pair consisting of a set $X$ and a group homomorphism $\rho : G \to S_X$.

**Remark 2.5.** Equivalent to the data of $\rho$ is the data of a map $a : G \times X \to X$ such that

- $a(1, x) = x$ for all $x \in X$.
- $a(g_1 g_2, x) = a(g_1, a(g_2, x))$ for all $g_1, g_2 \in G, x \in X$.

The relation is $\rho(g)(x) = a(g, x)$. One usually abbreviates $gx := a(g, x)$, so that $a$ is implicit.

**Remark 2.6.** The data of $\rho$ or $a$ as above is also said to be an **action of $G$ on $X$**. Usually $\rho$ or $a$ will be implicit for us in the notation (in the same way as when we speak of a group $G$, implicit is the multiplication operation $\times$).

**Example 2.7.** *Let* $V$ *be a finite-dimensional vector space over* $\mathbb{R}$*, equipped with an inner product* $\langle \cdot, \cdot \rangle$*. Let*

$$S := \{v \in V \mid ||v|| = 1\} \subset V.$$

*Let* $O(V) \subset GL(V)$ *be the subgroup consisting of the isometries - linear automorphisms* $T$ *satisfying* $\langle Tv_1, Tv_2 \rangle = \langle v_1, v_2 \rangle$ *for all* $v_1, v_2 \in V$*. Then Each* $T \in O(V)$ *preserves* $S$*, and we obtain an action of* $O(V)$ *on* $S$*, making* $S$ *an* $O(V)$*-set.*

**Example 2.8.** *Let* $H \subset G$ *be a subgroup. Then we can make* $G/H$ *to be a* $G$*-set by defining* $g(g'H) := gg'H$*.*

**Definition 2.9.** Let $X$ and $Y$ be $G$-sets. A **morphism of $G$-sets** (or **$G$-morphism**) $X \to Y$ is a map $f : X \to Y$ such that $f(gx) = gf(x)$ for all $g \in G, x \in X$.

**Notation 2.10.** Given $G$-sets $X$ and $Y$, we will denote by $\mathrm{Maps}_G(X, Y)$ the set of morphisms of $G$-sets from $X$ to $Y$.

**Remark 2.11.** One has the notion of isomorphism of $G$-sets - a morphism of $G$-sets which admits an inverse morphism of $G$-sets. One checks that a morphism of $G$-sets is an isomorphism of $G$-sets if and only if it is bijective.

**Exercise 2.1.** *Let $X$ be a $G$-set. Suppose that the action of $G$ on $X$ is transitive, meaning that $X$ is non-empty and for every $x_1, x_2 \in X$ there exists $g \in G$ such that $gx_1 = x_2$. Fix some $x_0 \in X$. Let*

$$Stab_G(x_0) := \{g \in G \mid gx_0 = x_0\} \subset G$$

*(this is called the **stabilizer** of $x_0$ in $G$). Then $Stab_G(x_0)$ is a subgroup of $G$. Show that we have naturally an isomorphism of $G$-sets $G/Stab_G(x_0) \cong X$.*

**Example 2.12.** *Let us return to our example of $O(V)$ acting on $S$. One checks that this action is transitive. Fix $s_0 := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in S$ and denote by $W \subset V$ the orthogonal complement to the span of $s_0$. Then $Stab_{O(V)}(s_0)$ is naturally isomorphic to $O(W)$. One has $O(V)/Stab_{O(V)}(s_0) \cong S$ as $O(V)$-sets, but one sometimes writes $O(V)/O(W) \cong S$ by slight abuse of notation (meaning that the reader should understand how $O(W)$ is realized as a subgroup of $O(V)$), or even simply $O(n)/O(n-1) \cong S^n$, where $n := \dim V$.*

### 2.1.3

Following a recurrent theme in mathematics, one makes a linear version of $G$-sets.

**Example 2.13.** *Consider the action of $S_2$ on $\{1, 2\}$ as above. It is transitive, so in some sense "indecomposable". But the action of $(12)$ squared is equal to the identity, so satisfies the equation $x^2 - 1 = 0$, and we want to exercise our reflex of breaking into the eigenspaces with eigenvalues $1$ and $-1$. Namely, one is tempted to consider formal linear combinations of the points $1$ and $2$ - so $\delta_1 + \delta_2$ will be an eigenvector for $(12)$ with eigenvalue $1$ and $\delta_1 - \delta_2$ will be an eigenvector for $(12)$ with eigenvalue $-1$. So linearization will allow us to decompose further a situation which seemed indecomposable.*

**Definition 2.14.** A **$G$-representation** (or a **representation of** $G$) is a pair consisting of a $k$-vector space $V$ and a group homomorphism $\rho : G \to GL(V)$. We again keep $\rho$ implicit usually. If $V$ and $W$ are $G$-representations, a **morphism of $G$-representations** (or **$G$-morphism**) $V \to W$ is a linear map

$T : V \to W$ such that $T(gv) = gT(v)$ for all $g \in G, v \in V$ (or in the more explicit $\rho$-language, we write $(V, \rho), (W, \pi)$ and then $T$ should satisfy $T \circ \rho(g) = \pi(g) \circ T$ for all $g \in G$).

**Notation 2.15.** Given $G$-representations $V$ and $W$, we will denote by $\mathrm{Hom}(V, W)$ the vector space of linear homomorphisms from $V$ to $W$, and by

$$\mathrm{Hom}_G(V, W) \subset \mathrm{Hom}(V, W)$$

the subspace of morphisms of $G$-representations.

**Remark 2.16.** Again we have the notion of an isomorphism of $G$-representations - a morphism of $G$-representations which admits an inverse morphism of $G$-representations. Again, one can check that a morphism of $G$-representations is an isomorphism of $G$-representations if and only if it is bijective.

**Example 2.17.** *We make $k^n$ a representation of $S_n$ by setting*

$$\sigma \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} := \begin{pmatrix} x_{\sigma^{-1}(1)} \\ \vdots \\ x_{\sigma^{-1}(n)} \end{pmatrix}.$$

**Example 2.18.** *More generally, let $X$ be a $G$-set. Consider the $k$-vector space $\mathrm{Fun}_k(X)$ of functions $X \to k$. We make $\mathrm{Fun}_k(X)$ a $G$-representation by setting*

$$(gf)(x) := f(g^{-1}x)$$

*(here $g \in G, f \in \mathrm{Fun}_k(X), x \in X$). Consider also the $k$-vector space $k[X]$ which has basis $(\delta_x)_{x \in X}$, where $\delta_x$ is a formal symbol created for each $x$. Then we make $k[X]$ a $G$-representation by associating to $g \in G$ the unique linear automorphism of $k[X]$ for which*

$$\delta_x \mapsto \delta_{gx}$$

*(here $g \in G, x \in X$).*

**Exercise 2.2.** *Let $X$ be a finite set. Then we have a natural isomorphism of $k$-vector spaces $\mathrm{Fun}_k(X) \cong k[X]$. If $X$ is a $G$-set, this isomorphism is an isomorphism of $G$-representations.*

**Example 2.19.** *Let $V$ be a $k$-vector space. The **trivial** $G$-representation structure on $V$ is given by associating to $g \in G$ the linear automorphism of $V$ which is the identity automorphism. The **trivial representation** of $G$ is $k$ together with the trivial $G$-representation structure.*

**Example 2.20.** *Let $\chi : G \to k^\times$ be a **character** (i.e. a homomorphism). Then we can make $k$ a $G$-representation by setting*

$$gc := \chi(g)c$$

*(here $g \in G, c \in k$). Let us denote this representation by $k_\chi$.*

**Exercise 2.3.** *Show that associating to a character $\chi : G \to k^\times$ the 1-dimensional $G$-representation $k_\chi$ sets a bijection between the set of characters $G \to k^\times$ and the set of isomorphism classes of 1-dimensional $G$-representations.*

## 2.2 Some notions and constructions

### 2.2.1

**Definition 2.21.** Let $V$ be a $G$-representation. A subspace $W \subset V$ is called a **$G$-subrepresentation** if $gw \in W$ for all $g \in G, w \in W$. In such a case, we can naturally view $W$ as a $G$-representation by itself (in the $\rho$-notation, we have $(V, \rho)$ and we now consider $(W, \rho_W)$ where $\rho_W$ is the linear automorphism of $W$ given by $\rho_W(g)(w) := \rho(g)(w)$).

**Example 2.22.** *Let $k^n$ be the representation of $S_n$ as above. Consider the subspace $W \subset k^n$ consisting of vectors* $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ *for which $x_1 + \ldots + x_n = 0$. Then $W$ is a $G$-subrepresentation of $k^n$.*

**Definition 2.23.** Let $V$ be a $G$-representation and $W \subset V$ a $G$-subrepresentation. Then we can consider the **quotient** $G$-representation $V/W$ - so the quotient vector space, with the $G$-action given by

$$g(v + W) := gv + W$$

(here $g \in G, v \in V$).

**Definition 2.24.** Let $V$ be a $G$-representation. We say that $V$ is **irreducible** (or **simple**) if $V \neq 0$ and for every $G$-subrepresentation $W \subset V$ we have either $W = 0$ or $W = V$. We say that $V$ is **indecomposable** if $V \neq 0$ and if $W_1, W_2 \subset V$ are $G$-subrepresentations such that $V = W_1 \oplus W_2$, then necessarily $W_1 = 0$ or $W_2 = 0$ (and then $W_1 = V$).

**Remark 2.25.** Clearly an irreducible representation is indecomposable. We will see later that if, for example, the characteristic of $k$ is zero, then the converse is also true.

**Example 2.26** (of an indecomposable representation which is not irreducible)**.** *Let us consider the action of $S_2$ on $k^2$ as above. Let us here consider the case when $k$ has characteristic 2. Then $k^2$ is not an irreducible $S_2$-representation, because we have the subrepresentation $W$ consisting of vectors whose sum of entries is zero. However, this is an indecomposable representation. Indeed, let $L$ be a 1-dimensional representation of $S_2$. Then $(12) \in S_2$ acts on $L$ by a scalar which squares to 1. Since $k$ has characteristic 2, this scalar is itself 1. Hence the $S_2$-action on $L$ is the trivial one. If $k^2$ would have been reducible, it would necessarily decompose as a direct sum $k^2 = W_1 \oplus W_2$ of two subrepresentations which are 1-dimensional. Then since each elements of $S_2$ acts trivially on $W_1$ and on $W_2$, it is clear that it acts trivially on the whole $k^2$. But (12) does not act trivially on $k^2$.*

**Definition 2.27.** Let $V_1, V_2$ be $G$-representations. We can construct the **direct sum** $G$-representation $V_1 \oplus V_2$. It is the direct sum of vector spaces, and the $G$-action is given by

$$g(v_1, v_2) := (gv_1, gv_2).$$

**Remark 2.28.** Of course, as it is already in linear algebra, the relation to the previous notion of direct sum is as follows. If $V$ is a $G$-representation and $W_1, W_2 \subset V$ are $G$-subrepresentations such that $V = W_1 \oplus W_2$, then we have a natural isomorphism between the just defined $W_1 \oplus W_2$ and $V$.

**Definition 2.29.** Let $V_1, V_2$ be $G$-representations, and let $T : V_1 \to V_2$ be a morphism of $G$-representations. We define the **kernel**

$$\mathrm{Ker}(T) := \{v_1 \in V_1 \mid T(v_1) = 0\}.$$

Then $\mathrm{Ker}(T)$ is a $G$-subrepresentation of $V_1$. We define the **image**

$$\mathrm{Im}(T) := \{v_2 \in V_2 \mid \exists v_1 \in V_1 \text{ s.t. } T(v_1) = v_2\}.$$

Then $\mathrm{Im}(T)$ is a $G$-subrepresentation of $V_2$. We define the **cokernel**

$$\mathrm{coKer}(T) := V_2/\mathrm{Im}(T).$$

## 2.3 Semisimplicity

### 2.3.1

**Definition 2.30.** Let $V$ be a $G$-representation. We say that $V$ is **semisimple** if for every $G$-subrepresentation $W \subset V$ there exists a $G$-subrepresentation $W' \subset V$ such that $V = W \oplus W'$.

**Remark 2.31.** As is known from linear algebra, for every $G$-subrepresentation $W \subset V$ (or any subspace $W \subset V$) there exists a subspace $W' \subset V$ such that $V = W \oplus W'$ (we say that $W'$ is a **complementary** subspace). So the matter here is in finding a complementary subspace which happens to be a $G$-subrepresentation.

**Remark 2.32.** Regarding the above distinction between irreducible and indecomposable representations, notice that clearly a semisimple indecomposable representation is irreducible. Example 2.26 provides an indecomposable representation which is not irreducible, so in particular not semisimple.

### 2.3.2

**Theorem 2.33** (Maschke's theorem)**.** *Suppose that $G$ is finite, and suppose that the characteristic of the field $k$ does not divide $|G|$. Then every finite-dimensional $G$-representation is semisimple.*

*First proof of Maschke's theorem.* Let $V$ be a finite-dimensional $G$-representation, it will be convenient to denote by $\rho : G \to GL(V)$ the implicit homomorphism. Let $W \subset V$ be a $G$-subrepresentation. As mentioned above, by linear algebra we know that there exists a subspace $W' \subset V$ such that $V = W \oplus W'$ (but $W'$ is not necessarily a $G$-subrepresentation). Let us consider the linear endomorphism $P : V \to V$ which is the projection on $W$ along $W'$. In other words,

$P$ is characterized by $P(w+w')=w$ whenever $w\in W, w'\in W'$. Let us now consider the following endomorphism $Q:V\to V$:

$$Q := \frac{1}{|G|}\sum_{g\in G}\rho(g)\circ P\circ\rho(g)^{-1}.$$

We claim that $Q$ is a projection operator on $W$. Indeed, first we check that $Q(w)=w$ for all $w\in W$. We have $\rho(g)^{-1}(w)=\rho(g^{-1})(w)\in W$ and therefore $P(\rho(g)^{-1}(w))=\rho(g)^{-1}(w)$ and therefore $\rho(g)(P(\rho(g)^{-1}(w)))=\rho(g)(\rho(g)^{-1}(w))=w$, so $Q(w)=\frac{1}{|G|}\sum_{g\in G}w=w$. Next, we notice that the image of $Q$ is contained in $W$ - this is clear as the image of $P$ is $W$ and the $\rho(g)$'s preserve $W$.

We now claim that $Q$ is a morphism of $G$-representations: For $h\in G$ we have

$$Q\circ\rho(h)=\left(\frac{1}{|G|}\sum_{g\in G}\rho(g)\circ P\circ\rho(g)^{-1}\right)\circ\rho(h)=\frac{1}{|G|}\sum_{g\in G}\rho(g)\circ P\circ\rho((h^{-1}g)^{-1})=\ldots$$

(we substitute $hg$ for $g$ in the sum)

$$\ldots=\frac{1}{|G|}\sum_{g\in G}\rho(hg)\circ P\circ\rho(g^{-1})=\rho(h)\circ\frac{1}{|G|}\sum_{g\in G}\rho(g)\circ P\circ\rho(g)^{-1}=\rho(h)\circ Q.$$

So, $Q:V\to V$ is a projection operator on $W$ which is also a morphism of $G$-representations. But then $V=W\oplus\mathrm{Ker}(Q)$ and $\mathrm{Ker}(Q)$ is a $G$-subrepresentation of $V$, so we get a complementary subrepresentation as desired. □

### 2.3.3

Let us conceptualize a bit the various ingredients of this proof.

**Definition 2.34.** Let $V$ and $W$ be $G$-representations. On the vector space $\mathrm{Hom}(V,W)$ of linear homomorphisms from $V$ to $W$ we define a structure of a $G$-representation by defining

$$(g\phi)(v) := g\phi(g^{-1}v)$$

(here $g\in G,\phi\in\mathrm{Hom}(V,W), v\in V$). Let us denote $g\star\phi$ for the result of acting by $g$ on $\phi$ in this representation, to not be confused with the composition $\rho(g)\circ\phi$.

**Definition 2.35.** Let $V$ be a $G$-representaiton. The subspace of **invariants** is defined as

$$V^G := \{v\in V \mid gv=v\ \forall g\in G\}\subset V.$$

Then $V^G$ is a $G$-subrepresentation of $V$.

**Definition 2.36.** Let $V$ be a $G$-representation and suppose that the characteristic of $k$ does not divide $|G|$. Define the **averaging operator** $Av_V^G : V \to V$ by:

$$Av_V^G(v) := \frac{1}{|G|} \sum_{g \in G} gv.$$

Then $Av_V^G$ is the identity on $V^G$, and also $\mathrm{Im}(Av_V^G) = V^G$. In other words, $Av_V^G$ is a projection operator on $V^G$.

**Remark 2.37.** Notice that $T \in \mathrm{Hom}(V, W)$ is a morphism of $G$-representations if and only if $g \star T = T$ for all $g \in G$. In other words,

$$\mathrm{Hom}_G(V, W) = \mathrm{Hom}(V, W)^G.$$

**Remark 2.38.** Thus, in the above proof, we consider the $G$-representation $\mathrm{Hom}(V, V)$ and the element $P \in \mathrm{Hom}(V, V)$. We then consider $Q := Av^G_{\mathrm{Hom}(V,V)}(P)$. Then $Q \in \mathrm{Hom}(V, V)^G = \mathrm{Hom}_G(V, V)$ and we also checked that if $P$ was a projection operator on a $G$-subrepresentation $W$, then $Q$ will also be a projection operator on $W$.

### 2.3.4

*Second proof of Maschke's theorem.* This proof works when $k = \mathbb{R}$ or $k = \mathbb{C}$. Let us consider an arbitrary inner product $\langle -, - \rangle_0$ on $V$. We then define a new inner product by

$$\langle v_1, v_2 \rangle := \frac{1}{|G|} \sum_{g \in G} \langle gv_1, gv_2 \rangle_0.$$

The inner product $\langle -, - \rangle$ is then **$G$-invariant**, in the sense that, for $h \in G$:

$$\langle hv_1, hv_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \langle ghv_1, ghv_2 \rangle_0 = \dots$$

(we substitute $gh^{-1}$ for $g$ in the sum)

$$\dots = \frac{1}{|G|} \langle gv_1, gv_2 \rangle_0 = \langle v_1, v_2 \rangle.$$

Let $U$ be the orthogonal complement, with repsect to $\langle -, - \rangle$, to $W$ in $V$. So $V = W \oplus U$, and it is left to show that $U$ is a $G$-subrepresentation of $V$. Indeed, if $u \in U$ and $g \in G$ then for any $w \in W$ we have

$$\langle w, gu \rangle = \langle gg^{-1}w, gu \rangle = \langle g^{-1}w, u \rangle = 0$$

(since $g^{-1}w \in W$). Hence $gu$ is perpendicular to $W$ with respect to $\langle -, - \rangle$, i.e. $gu \in U$. □

**Remark 2.39.** In the proof appeared he theme of positivity. Notice that it is clear that the average of a positive-definite bilinear form is positive-definite, and in particular non-degenerate. However, it is not true that the average of a non-degenerate form is necessarily non-degenerate. On the other hand, of course, positive structures are only available when working over a very restricted class of fields (such as $\mathbb{R}$ and $\mathbb{C}$).

#### 2.3.5

*Third proof of Maschke's theorem.* Consider the projection $p : V \to V/W$. We want to show that there exists a $G$-morphism $\iota : V/W \to V$ such that $p \circ \iota = \mathrm{Id}_{V/W}$ (then the image of $\iota$ will be a $G$-subrepresentation of $V$ which is complementary to $W$). Stating more generally, given a surjective morphism of $G$-representations $p : V \to Z$, and a $G$-representation $U$, we would like to show that $\mathrm{Hom}_G(U, V) \to \mathrm{Hom}_G(U, Z)$ (given by composing with $p$) is surjective. This map is the restriction of the more general map $\mathrm{Hom}(U, V) \to \mathrm{Hom}(U, Z)$ given by composition with $p$. The latter map, one immediately checks, is a morphism of $G$-representations, and it is surjective by linear algebra. We therefore recast the problem as follows: Given $G$-representations $V$ and $W$ and a surjective $G$-morphism $p : V \to W$, the induced morphism $V^G \to W^G$ (obtained by restriction) is surjective as well. Indeed, let $w \in W^G$. Let $v \in V$ be such that $p(v) = w$. Then $p(Av_V^G(v)) = Av_W^G(p(v)) = Av_W^G(w) = w$, so $Av_V^G(v)$ is a preimage of $w$ under our $V^G \to W^G$. □

## 2.4 Decomposition into irreducibles

We suppose throughout this subsection that $G$ is finite and the characteristic of $k$ does not divide $|G|$.

#### 2.4.1

**Lemma 2.40.** *Let $V$ be a finite-dimensional $G$-representation. Then there exist irreducible $G$-representations $E_1, \ldots, E_r$ such that $V \cong E_1 \oplus \ldots \oplus E_r$ (as $G$-representaitons).*

*Proof.* The proof is by induction on the dimension of $V$. If $\dim V = 0$ then the empty family of $E_i$'s ($r := 0$) will do. Now for the induction step, if $V$ is irreducible then $r := 1, E_1 := V$ will do. If $V$ is not irreducible, we can find a $G$-subrepresentation $W \subset V$ such that $W \neq 0$ and $W \neq V$. By Maschke's theorem, we can find a $G$-subrepresentation $U \subset V$ such that $V = W \oplus U$. Then $U \neq V$ since $W \neq 0$, and so by the induction hypothesis we can find irreducible $G$-representations $E_1, \ldots, E_r$ and $F_1, \ldots, F_m$ such that

$$W \cong E_1 \oplus \ldots \oplus E_r, \quad U \cong F_1 \oplus \ldots \oplus F_m.$$

Then

$$V \cong E_1 \oplus \ldots \oplus E_r \oplus F_1 \oplus \ldots \oplus F_m.$$

□

### 2.4.2

We now naturally would like to understand the uniqueness in the above decomposition. For that, we first show:

**Lemma 2.41** (Schur's lemma)**.** *Let $E$ and $F$ be irreducible $G$-representations. Let $T : E \to F$ be a morphism of $G$-representations. Then either $T = 0$ or $T$ is an isomorphism. In particular, if $E$ and $F$ are not isomorphic, then* $\mathrm{Hom}_G(E, F) = 0$.

*Proof.* Let $T : E \to F$ be a non-zero morphism of $G$-representations - we want to show that $T$ is an isomorphism. Consider $\mathrm{Ker}(T) \subset E$ (recall that it is a $G$-subrepresentation). Since $T$ is non-zero, $\mathrm{Ker}(T) \neq E$. But since $E$ is irreducible, this implies $\mathrm{Ker}(T) = 0$. So $T$ is injective. Now consider $\mathrm{Im}(T) \subset F$ (recall that it is a $G$-subrepresentation). Since $T$ is non-zero, $\mathrm{Im}(T) \neq 0$. But since $F$ is irreducible, this implies $\mathrm{Im}(T) = F$. So $T$ is surjective. Thus, $T$ is injective and surjective, so bijective, and hence an isomorphism of $G$-representations. □

### 2.4.3

A small "by the way":

**Exercise 2.4.** *Show that every irreducible $G$-representation is finite-dimensional (recall that now we assume that $G$ is finite). So we don't need to say "finite-dimensional irreducible $G$-representation" each time, and can simply say "irreducible $G$-representation" without ambiguity.*

**Claim 2.42.** *Let $V$ be a finite-dimensional $G$-representation. Let*

$$E_1, \ldots, E_r$$

*and*

$$F_1, \ldots, F_m$$

*be irreducible $G$-representations such that*

$$V \cong E_1 \oplus \ldots \oplus E_r$$

*and*

$$V \cong F_1 \oplus \ldots \oplus F_m.$$

*Then for every irreducible $G$-representation $E$, the number of $1 \leq i \leq r$ for which $E_i$ is isomorphic to $E$ is equal to the number of $1 \leq j \leq m$ for which $F_j$ is isomorphic to $E$, both numbers being equal to*

$$\frac{\dim \mathrm{Hom}_G(V, E)}{\dim \mathrm{Hom}_G(E, E)}.$$

*Proof.* Let us denote $d := \dim \mathrm{Hom}_G(E, E)$ (of course, $d \in \mathbb{Z}_{\geq 1}$ since at least the identity endomorphism is a $G$-morphism). For an irreducible $G$-representation

$F$, $\dim \operatorname{Hom}_G(E, F)$ is equal to 0 if $F$ is not isomorphic to $E$ (by Schur's lemma) and to $d$ if $F$ is isomorphic to $E$. Therefore:

$$\dim \operatorname{Hom}_G(V, E) = \dim (\operatorname{Hom}_G(E_1 \oplus \ldots \oplus E_r, E)) =$$

$$= \dim (\operatorname{Hom}_G(E_1, E) \oplus \ldots \operatorname{Hom}_G(E_r, E)) =$$

$$= \dim \operatorname{Hom}_G(E_1, E) + \ldots + \dim \operatorname{Hom}_G(E_r, E) =$$

$$= d \cdot |\{1 \leq i \leq r \mid E_i \text{ isomorphic to } E\}|.$$

Thus we obtain

$$|\{1 \leq i \leq r \mid E_i \text{ isomorphic to } E\}| = \frac{\dim \operatorname{Hom}_G(V, E)}{d}$$

and the claim follows. □

**Definition 2.43.** Let $V$ be a finite-dimensional $G$-representation and $E$ an irreducible $G$-representation. To define the **multiplicity** of $E$ in $V$, denoted $[V : E]$, write $V \cong E_1 \oplus \ldots \oplus E_r$ where the $E_i$'s are irreducible $G$-representaitons and then the multiplicity is set to be the number of $1 \leq i \leq r$ for which $E_i$ is isomorphic to $E$. By the above claim, this does not depend on the choice.

**Remark 2.44.** A "more correct" approach to multiplicity, which would work for finite-dimensional representation of any group over any field, is to consider Jordan-Holder series for the representation and the resulting subquotients.

### 2.4.4

Let us here also briefly introduce isotypic components. Given a finite-dimensional $G$-representations $V$ and an irreducible $G$-representation $E$, we consider the subrepresentation of $V$ which is the sum of all irreducible subrepresentations isomorphic to $E$. This is the **isotypic component** $V_E$.

**Lemma 2.45.** *Write $V = E_1 \oplus \ldots \oplus E_r$ where the $E_i$'s are irreducible subrepresentations of $V$. Then $V_E$ is equal to the sum of the $E_i$'s which are isomorphic to $E$.*

*Proof.* Clearly the sum of the $E_i$'s which are isomorphic to $E$ is contained in $V_E$. To show the inclusion in the other direction, let $F \subset V$ be an irreducible subrepresentation which is isomorphic to $E$. We need to show that $F$ is contained in the sum of the $E_i$'s which are isomorphic to $E$. For this, it is enough to show that given an $E_i$ which is not isomorphic to $E$, the composition of the inclusion $F \to V$ with the projection $V \to E_i$ (arising from the direct sum decomposition $V = E_1 \oplus \ldots \oplus E_r$) is zero. But this composition is a $G$-morphism $F \to E_i$ between irreducible $G$-representations, and therefore, by Schur's lemma, it is either an isomorphism or zero. Since $F$ and $E_i$ are not isomorphic, it must be zero, as desired. □

**Remark 2.46.** We thus see that although a decomposition of $V$ into a direct sum of irreducible representations is not unique, if we gather all the irreducible representations appearing in the decomposition according to isomorphism, we do obtain a decomposition (into isotypic components) which does not depend on any choice.

**Exercise 2.5.** *Let $V$ be a finite-dimensional $G$-representation and let $E$ be an irreducible $G$-representation. Show that there exists a unique $G$-morphism $V \to V$ which is a projection onto $V_E$. Equivalently, there exists a unique $G$-subrepresentation of $V$ which is complementary to $V_E$. When we decompose $V$ as a direct sum of irreducible $G$-subrepresentations, describe this complementary subrepresentation as the direct sum of all the irreducible summands in the chosen decomposition which are not isomorphic to $E$.*

**Exercise 2.6.** *Let $V$ be a finite-dimensional $G$-representation. We had the projection $Av_V^G$ onto the subspace $V^G$ of $G$-invariants, which is a $G$-morphism. Notice that $V^G$ can be interpreted as $V_{k_1}$, the isotypic component of $V$ corresponding to the trivial irreducible representation. Use the previous exercise to understand what the kernel of $Av_V^G$ is. Let us note that we therefore get a concrete formula for the $G$-morphic projection onto the isotypic component corresponding to the trivial representation. One can ask whether there are similar formulas for othe irreducible representations. We will later see that the theory of characters provides such formulas.*

### 2.4.5

**Lemma 2.47** (Schur's lemma, continued)**.** *Suppose that $k$ is algebraically closed. Let $E$ be an irreducible $G$-representation. Then $\mathrm{End}_G(E) := \mathrm{Hom}_G(E, E)$ is equal to $k \cdot \mathrm{Id}_E$.*

*Proof.* Let $T \in \mathrm{End}_G(E)$. Since $k$ is algebraically closed, $T$ admits an eigenvalue $c \in k$, i.e. $T - c \cdot \mathrm{Id}_E$ is not invertible. But $T - c \cdot \mathrm{Id}_E$ is a $G$-endomorphism, and therefore by Schur's lemma $T - c \cdot \mathrm{Id}_E = 0$, i.e. $T = c \cdot \mathrm{Id}_E$. □

**Corollary 2.48.** *Suppose that $k$ is algebraically closed. Let $V$ be a finite-dimensional $G$-representation and $E$ an irreducible $G$-representation. Then*

$$[V : E] = \dim \mathrm{Hom}_G(V, E).$$

*Proof.* We saw above that

$$[V : E] = \frac{\dim \mathrm{Hom}_G(V, E)}{\dim \mathrm{End}_G(E)}$$

and we saw just now that $\mathrm{End}_G(E) = k \cdot \mathrm{Id}_E$ so $\dim \mathrm{End}_G(E) = 1$. □

### 2.4.6 The regular representation

**Definition 2.49.** The **regular $G$-set** is $G$ with the standard $G$-action by multiplication: $\rho(g)(h) := gh$ where $g, h \in G$. The **regular $G$-representation** is obtained from this by consdering $k[G]$ as above. Thus, $k[G]$ has basis $(\delta_h)_{h\in G}$ and the structure of $G$-representation on it is by $\rho(g)(\delta_h) = \delta_{gh}$ where $g, h \in G$.

**Remark 2.50.** Recall from linear algebra that given a set $X$ and a vector space $V$, we have an isomorphism of vector spaces

$$\mathrm{Hom}(k[X], V) \cong \mathrm{Maps}(X, V)$$

where on the left we consider linear homomorphisms and on the left we considers arbitrary maps, and the vector space structures are the natural ones, by adding and multiplying by scalar in $V$. The isomorphism is obtained by sending $T : k[X] \to V$ to $f : X \to V$ given by $f(x) := T(\delta_x)$.

**Lemma 2.51.** *Let $X$ be a $G$-set and $V$ a $G$-representation. The above isomorphism of vector spaces*

$$\mathrm{Hom}(k[X], V) \cong \mathrm{Maps}(X, V)$$

*restricts to an isomorphism of vector spaces*

$$\mathrm{Hom}_G(k[X], V) \cong \mathrm{Maps}_G(X, V).$$

*Proof.* Left as an exercise. □

**Proposition 2.52.** *Let $E$ be an irreducible $G$-representation. Then the multiplicity $[k[G] : E]$ of $E$ in the regular representation of $G$ is equal to $\frac{\dim E}{\dim \mathrm{End}_G(E)}$. In particular, if $k$ is algebraically closed, then the multiplicity $[k[G] : E]$ is equal simply to $\dim E$.*

*Proof.* We have

$$[k[G] : E] = \frac{\dim \mathrm{Hom}_G(k[G], E)}{\dim \mathrm{End}_G(E)} = \frac{\dim \mathrm{Maps}_G(G, E)}{\dim \mathrm{End}_G(E)} = \ldots$$

(notice that we have a natural isomorphism of vector spaces $\mathrm{Maps}_G(G, E) \cong E$ given by sending $f$ to $f(1)$)

$$\ldots = \frac{\dim E}{\dim \mathrm{End}_G(E)}.$$

□

**Corollary 2.53.** *There are finitely many isomorphism classes of irreducible $G$-representations.*

*Proof.* From the above, every irreducible $G$-representations appears with non-zero multiplicity in the regular $G$-representation, but as the regular $G$-representation is finite-dimensional, clearly only finitely many irreducible $G$-representations, up to isomorphism, have non-zero multiplicity in it. □

**Corollary 2.54** (basic formula)**.** *Suppose that $k$ is algebraically closed. Let $E_1, \ldots, E_r$ be a list of all the irreducible $G$-representations, each appearing once, up to isomorphism. Then*

$$|G| = (\dim E_1)^2 + \ldots + (\dim E_r)^2.$$

*Proof.* We have

$$|G| = \dim k[G] = \sum_{1 \leq i \leq r} [k[G] : E_i] \cdot \dim E_i = \sum_{1 \leq i \leq r} \dim E_i \cdot dim E_i.$$

□

#### 2.4.7

Let us consider the example of the group $S_3$. We assume that $k$ is an algebraically closed field of characteristic not dividing $|S_3|$, i.e. not 2 or 3. We have two characters $S_3 \to k^\times$ and therefore two 1-dimensional representations of $S_3$, up to isomorphism - the trivial representation and the sign representation corresponding to $sgn : S_3 \to \{\pm 1\} \subset k^\times$. We have therefore

$$6 = |S_3| = 1^2 + 1^2 + ?^2 + \ldots.$$

We see that the only possibility is that there is one more irreducible representation, of dimension 2. To find it, we can consider our $k^3$ with the usual permutation representation of $S_3$. As mentioned above, we have a 2-dimensional subrepresentation $W \subset k^3$ consisting of the vectors the sum of whose entries is equal to 0. One checks that this is an irreducible representation (left as an exercise).

### 2.5 The group algebra

#### 2.5.1

For us, a ring will be possibly non-commutative, but always with 1.

**Definition 2.55.** A $k$-algebra is a ring $A$ which is also a $k$-vector space, such that the multiplication $A \times A \to A$ is $k$-bilinear.

**Remark 2.56.** Let $A$ be a $k$-algebra. Then we have a map $k \to A$ given by $c \mapsto c \cdot 1$ (where 1 is the unit in the ring $A$). This map is injective, so we can think of $k$ as sitting inside $A$. In fact, we can equivalently define a $k$-algebra as a ring $A$ together with a ring homomorphism $k \to Z(A)$, where $Z(A)$ is the center of $A$.

### 2.5.2

**Definition 2.57.** Let $R$ be a ring. A (left) **$R$-module** is an abelian group $M$ together with an action map $R \times M \to M$ (we again abbreviate $rm$ for the result of applying this action map to $(r, m)$), such that

- $1m = m$ for all $m \in M$.
- $r(m_1 + m_2) = rm_1 + rm_2$ for all $r \in R$ and $m_1, m_2 \in M$.
- $(r_1 + r_2)m = r_1 m + r_2 m$ for all $r_1, r_2 \in R$ and $m \in M$.
- $(r_1 r_2)m = r_1(r_2 m)$ for all $r_1, r_2 \in R$ and $m \in M$.

**Remark 2.58.** By an $R$-module, unless specified otherwise, we will mean a left $R$-module.

**Definition 2.59.** Let $R$ be a ring and let $M, N$ be $R$-modules. A **morphism of $R$-modules** (or **$R$-morphism**) from $M$ to $N$ is a abelian group homomorphism $T : M \to N$ such that $T(rm) = rT(m)$ for all $r \in R, m \in M$.

**Remark 2.60.** We skip the explicit repetition of notions such as submodules, direct sums, kernels, etc. These are defined as in previous frameworks.

**Remark 2.61.** Let now $A$ be a $k$-algebra. Let $M$ be an $A$-module. Then $M$ becomes naturally a $k$-vector space, since $k \hookrightarrow Z(A) \subset A$. But also, a lot of times we are already given a $k$-vector space $M$, and then if we give $M$ an $A$-module structure, or suppose given to $M$ an $A$-module structure, we will always implicitly mean that the given $k$-vector space structure of $M$ and that obtained from the $A$-module structure are the same (unless specified otherwise in rare cases). In other words, if $M$ carries its own $k$-vector space structure and is an $A$-module, the action map $A \times M \to M$ is always supposed to be $k$-bilinear.

### 2.5.3

**Definition 2.62.** The group algebra of $G$, denoted $k[G]$ is, as a $k$-vector space, $k[G]$ as above (i.e. the vector space whose basis is $(\delta_g)_{g \in G}$), with the ring structure characterized as the unique $k$-bilinear pairing $k[G] \times k[G] \to k[G]$ sending $(\delta_g, \delta_h)$ to $\delta_{gh}$.

**Lemma 2.63.** *Let $A$ be a $k$-algebra. Then we have a natural bijection*

$$\mathrm{Hom}(k[G], A) \cong \mathrm{Hom}(G, A^\times).$$

*Here on the left we consider morphisms of $k$-algebras. On the right we consider morphisms of groups. The bijection is given by sending $\alpha$ on the left to $g \mapsto \alpha(\delta_g)$ on the right.*

*Proof.* Left as an exercise. $\square$

**Corollary 2.64.** *Let $V$ be a $k$-vector space. Then we have a natural bijection between structures of a $G$-representation on $V$ and of a $k[G]$-module on $V$.*

*Proof.* We interpret $G$-representation structures on $V$ as group homomorphisms $G \to \text{End}(V)^\times = \text{GL}(V)$. We interpret $k[G]$-module structures on $V$ as $k$-algebra homomorphisms $k[G] \to \text{End}(V)$. We now use the previous lemma. $\square$

**Exercise 2.7.** *Let $V, W$ be two $G$-representations, so also $k[G]$-modules as explained above. Then a linear map $T : V \to W$ is a morphism of $G$-representations if and only if it is a morphism of $k[G]$-modules.*

**Remark 2.65.** To summarize, $k[G]$ is the "linear envelope" of $G$, and to study $G$-representations over $k$ is the same as to study $k[G]$-modules.

## 2.6 The non-commutative Fourier transform

We assume throughout this subsection that $G$ is finite and the characteristic of $k$ does not divide $|G|$.

### 2.6.1

**Definition 2.66.** A ring $R$ is said to be a **division ring** if for every $r \in R$, if $r \neq 0$ then $r$ is invertible in $R$. A $k$-algebra $A$ is said to be a **division algebra** if it is, as a ring, a division ring.

We can restate part of Schur's lemma as follows:

**Lemma 2.67** (Schur's lemma, continuation)**.** *Let $E$ be an irreducible $G$-representation. Then $\text{End}_G(E)$ is a division algebra (the multiplication giving it an algebra structure is of course composition of linear endomorphisms).*

*Proof.* This immediately follows from Schur's lemma, as stated above. $\square$

Let us, although it is not strictly necessary at this point, deduce Lemma 2.47 from Lemma 2.67. First, we have the following lemma:

**Lemma 2.68.** *Let $A$ be a finite-dimensional division $k$-algebra, and $B \subset A$ a $k$-subalgebra. Then $B$ is also a division $k$-algebra.*

*Proof.* Let $b \in B$ and suppose that $b \neq 0$. We want to show that $b$ is invertible in $B$. Let $m \in k[x]$ be the minimal polynomial of the linear map $A \to A$ given by $a \mapsto ba$. Then $m(b) = 0$ (check!). We write $m(x) = xn(x) + c$ (where $n \in k[x]$ and $c \in k$) and notice that $c \neq 0$, because the linear map $a \mapsto ba$ is invertible (as $b$ has an inverse in $A$). Therefore $0 = m(b)$ can be rewritten as $b \cdot (-c^{-1}n(b)) = 1$ and of course also $(-c^{-1}n(b)) \cdot b = 1$, so $-c^{-1}n(b)$ is an inverse of $b$ in $B$. $\square$

Next, we have the following claim:

**Claim 2.69.** *Suppose that $k$ is algebraically closed, and let $A$ be a finite-dimensional division $k$-algebra. Then $A = k$.*

*Proof.* Let $a \in A$. Denote by $B \subset A$ the $k$-subalgebra generated by $a$ (this is the linear span of $1, a, a^2, \ldots$). Then by the above lemma, $B$ is a division $k$-algebra itself. However, $B$ is also commutative. Therefore $B$ is a field. $B$ is a finite extension field of $k$, so since $k$ is algebraically closed, we have $B = k$. Therefore $a \in k$. As $a$ was arbitrary, we obtain $A = k$. □

Finally, we can deduce Lemma 2.47 from Lemma 2.67. The latter shows that $\mathrm{End}_G(E, E)$ is a division $k$-algebra. Since it is finite-dimensional, we just saw that, assuming that $k$ is algebraically closed, we have $\mathrm{End}_G(E, E) = k$.

### 2.6.2

Modules over division rings behave similarly to modules over a field (i.e. vector spaces). But one of course has to be careful that there are left modules and right modules (as we said, when we say "module" we mean implicitly "left module"). In particular, if $D$ is a division ring and $V$ is a $D$-module, then there exists a family $(e_i)_{i\in I}$ of non-zero elements in $V$ such that

$$V = \oplus_{i\in I} D \cdot e_i$$

(we say then that $(e_i)_I$ is a basis for $V$ over $D$). If $V$ is finitely generated over $D$, then $I$ must be finite. More generally, if $W \subset V$ is a $D$-submodule and we are already given a basis $(e_i)_{i\in I}$ of $W$, we can find a family $(e_i)_{i\in J}$ of elements in $V$ such that $(e_i)_{i\in I \coprod J}$ is a basis of $V$. The proofs are the same as in the case when $D$ is a field. One also shows in the same way as for fields that $|I|$ is independent of the choices, and this is called the **dimension** of $V$ over $D$. If $D$ is a $k$-algebra, we have the formula (usually the student sees it when discussing towers of field extensions, but the proof is the same):

$$\dim_k V = \dim_D V \cdot \dim_k D.$$

### 2.6.3

Let $R$ be a ring and $M$ an $R$-module. Let us denote $S := \mathrm{End}_R(M)$. Then $S$ is a ring, and we can consider $M$ as an $S$-module. We have a canonical ring homomorphism $R \to \mathrm{End}_S(M)$, simply given by the $R$-module structure on $M$. Thus, $S$ consists of endomorphisms of the abelian group $M$ which commute with those coming from $R$, and therefore, tautologically, the endomorphisms that come from $R$ commute with those from $S$.

### 2.6.4

Let $E$ be an irreducible $G$-representation. We denote $D_E := \mathrm{End}_G(E)$ (recall that $D_E$ is a division $k$-algebra by Schur's lemma). Then as just explained, we have a natural $k$-algebra morphism

$$\mathcal{F}_E : k[G] \to \mathrm{End}_{D_E}(E).$$

### 2.6.5

Let us denote by

$$E_1, \dots, E_r$$

an exhaustive list of irreducible $G$-representations (by exhaustive we mean that the representations in the list are pairwise non-isomorphic and that every irreducible $G$-representation is isomorphic to one in the list). We gather the $\mathcal{F}_{E_i}$'s into one $k$-algebra morphism

$$\mathcal{F} : k[G] \to \prod_{1 \le i \le r} \mathrm{End}_{D_{E_i}}(E_i).$$

One might call $\mathcal{F}$ the **non-commutative Fourier transform** (we will try later to see why this name).

**Proposition 2.70** (A case of the Artin-Wedderburn theorem)**.** *The $k$-algebra homomorphism $\mathcal{F}$ is an isomorphism.*

*Proof.* We will show that $\mathcal{F}$ is an isomorphism by showing that it is injective and that the source and target have the same dimension over $k$.

First let us show that $\mathcal{F}$ is injective. Let $a \in k[G]$ and suppose that $\mathcal{F}(a) = 0$. This means that $a$ acts by zero on every irreducible $G$-representation. But then it acts by zero on every finite-dimensional $G$-representation, as every finite-dimensional $G$-representation is a direct sum of irreducible $G$-representations. In particular, $a$ acts by zero on $k[G]$, the regular representation. This means $ab = 0$ for all $b \in k[G]$, and so $a = a \cdot 1 = 0$.

Now we want to show that the dimensions over $k$ of the source and target of $\mathcal{F}$ match. The dimension of the source is $|G|$. Let us abbreviate $e_i := \dim_k E_i$ and $d_i := \dim_k D_{E_i}$. Given finitely generated modules $E$ and $F$ over a division $k$-algebra $D$ of finite-dimension, we have

$$\dim_k \mathrm{Hom}_D(E, F) = \dim_D E \cdot \dim_k F = \frac{\dim_k E}{\dim_k D} \cdot \dim_k F = \frac{\dim_k E \cdot \dim_k F}{\dim_k D}.$$

Indeed, choosing a basis $v_1, \dots, v_r$ of $E$ over $D$, we can construct an isomorphism of $k$-vector spaces

$$\mathrm{Hom}_D(E, F) \xrightarrow{\sim} F^{\oplus r}$$

by sending $T$ to $(T(v_1), \dots, T(v_r))$. We thus obtain:

$$\dim_k \mathrm{End}_{D_{E_i}}(E_i) = \frac{(\dim_k E_i)^2}{\dim_k D_{E_i}} = e_i^2 / d_i.$$

Therefore we need to show that

$$|G| = \sum_{1 \le i \le r} \frac{e_i^2}{d_i}.$$

And indeed:

$$|G| = \sum_{1\le i\le r} [k[G] : E_i] \cdot \dim_k E_i = \sum_{1\le i\le r} \frac{e_i}{d_i} \cdot e_i$$

where the formula $[k[G] : E_i] = e_i/d_i$ we have seen in Proposition 2.52. □

**Remark 2.71.** Suppose that $k$ is algebraically closed (this is the most important case). Then the $D_{E_i}$'s are all equal to $k$, and we obtain an isomorphism of $k$-algebras

$$\mathcal{F} : k[G] \xrightarrow{\sim} \prod_{1\le i\le r} \mathrm{End}_k(E_i).$$

One says: The group algebra is a product of matrix algebras.

### 2.6.6

Let us extract the commutative consequence of the non-commutative Fourier transform, by passing to the centers of the $k$-algebras which are the domain and target of $\mathcal{F}$.

**Exercise 2.8.** *Identifying $k[G]$ with $\mathrm{Fun}_k(G)$ (by sending $f \in \mathrm{Fun}_k(G)$ to $\sum_{g\in G} f(g)\cdot\delta_g \in k[G]$), the center $Z(k[G])$ gets identifies with the subspace $\mathrm{Fun}_k(G)^{cl} \subset \mathrm{Fun}_k(G)$ consisting of functions $f$ for which $f(hgh^{-1}) = f(g)$ for all $g, h \in G$ (equivalently, $f(gh) = f(hg)$ for all $g, h \in G$) - this subspace is called the subspace of* ***class functions****, and we will return to it later, when discussing character theory.*

**Exercise 2.9.** *Let $D$ be a division ring and $V$ a finite-dimensional $D$-module. Then the morphism of rings $Z(D) \to Z(\mathrm{End}_D(V))$ (given by sending $z$ to the endomorphism of $V$ given by $v \mapsto zv$) is an isomorphism. To show this, we can first understand that, once we choose a basis for $V$ over $D$, we can construct an isomorphism of $\mathrm{End}_D(V)$ sith the ring of $(n\times n)$-matrices over $D^{op}$, where $n := \dim_D V$ and $D^{op}$ denotes the ring opposite to $D$ (i.e. same set, same addition and multiplication is given by $d_1 \overset{\text{new}}{*} d_2 := d_2 \overset{\text{old}}{*} d_1$). Then it is an easy exercise to show that a matrix which commutes with all others is necessarily scalar, and furthermore, this scalar in $D^{op}$ must in fact lie in the center. Note that $Z(D) \cong Z(D^{op})$.*

**Corollary 2.72** (basic formula)**.** *Suppose that $k$ is algebraically closed. The cardinality of the set of isomorphism classes of irreducible $G$-representations is equal to the cardinality of the set of conjugacy classes in $G$.*

*Proof.* By Exercise 2.8, the dimension of the center of the domain of $\mathcal{F}$ is the number of conjugacy classes in $G$. By Exercise 2.8, the dimension of the center of the target of $\mathcal{F}$ is, in notation from above, $\sum_{1\le i\le r} \dim Z(D_{E_i})$. This holds in general, but if $k$ is also algebraically closed as we assume now, then $D_{E_i} = k$ and the last sum becomes equal to $r$, the number of irreducible $G$-representations, up to isomorphism. □

**Remark 2.73.** Thus, let us repeat the two most basic numeric facts. Given a finite group $G$ and an algebraically closed field $k$ whose characteristic does not divide $|G|$, and denoting by $d_1, \ldots, d_r$ the dimensions of the various irreducible $G$-representations over $k$ (up to isomorphism), we have that $r$, the number of irreducible $G$-representations, is equal to the number of conjugacy classes in $G$, while $\sum_{1 \leq i \leq r} d_i^2$ is equal to the number of elements in $G$.

**Remark 2.74.** Let us mention an interesting extrapolation of the two basic numeric facts (although we will not need it at all), due to Frobenius. We define the "zeta function" of $G$ to be

$$\zeta_G(s) := \sum_{1 \leq i \leq r} d_i^{-s}.$$

Then we saw that $\zeta_G(0)$ is equal to the number of conjugacy classes in $G$, while $\zeta_G(-2)$ is equal to the number of elements in $G$. One has

$$\zeta_G(-2+2n) = \frac{1}{|G|^{2n-1}} |c_n^{-1}(1)|$$

for $n \in \mathbb{Z}_{\geq 0}$, where $c_n : G^{2n} \to G$ is given by

$$c_n(x_1, y_1, \ldots, x_n, y_n) := [x_1, y_1] \cdot \ldots \cdot [x_n, y_n].$$

### 2.6.7

Given an irreducible $G$-representation $E$, there is a unique element $e_E \in k[G]$ which acts by identity on $E$ and by 0 on any irreducible $G$-representation which is not isomorphic to $E$. Indeed, this is clearly simply the element for which $\mathcal{F}(e_E) = (T_i)_{1 \leq i \leq r}$ where $T_i = \mathrm{Id}_{E_i}$ for the $i$ for which $E_i$ is isomorphic to $E$, and $T_i = 0$ for other $i$'s. Notice that $e_E$ is a central idempotent in $k[G]$, i.e. $e_E \in Z(k[G])$ and $e_E^2 = e_E$. Notice that the action by $e_E$ on any finite-dimensional $G$-representation $V$ is the unique $G$-morphic projection onto the isotypic component $V_E$ we talked about before. Thus the question we had before, about a formula for this projection, can be restated as asking a formula for $e_E$, i.e. to describe the scalars $c_g$ such that $e_E = \sum_{g \in G} c_g \cdot \delta_g$. As we said, we will return to this when we study character theory.

### 2.6.8

Let us consider again $S_3$ (and work over $\mathbb{C}$). Choosing a basis for the standard representation $E := \{(x_1, x_2, x_3)^t \mid x_1 + x_2 + x_3 = 0\}$ of before, we obtain an isomorphism of $\mathbb{C}$-algebras

$$\mathbb{C}[S_3] \cong \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$$

corresponding to the exhaustive list of irreducible representations $\mathbb{C}_1$, $\mathbb{C}_{\mathrm{sgn}}$ and $E$. We will not do it now, but one can see that $(1,0,0)$ on the right corresponds to

$$e_{\mathbb{C}_1} = \frac{1}{6} \cdot \delta_{\mathrm{id}} + \frac{1}{6} \cdot \delta_{(123)} + \frac{1}{6} \cdot \delta_{(132)} + \frac{1}{6} \cdot \delta_{(12)} + \frac{1}{6} \cdot \delta_{(23)} + \frac{1}{6} \cdot \delta_{(13)}$$

on the left, $(0,1,0)$ on the right corresponds to

$$e_{\mathbb{C}_{\text{sgn}}} = \frac{1}{6}\cdot\delta_{\text{id}} + \frac{1}{6}\cdot\delta_{(123)} + \frac{1}{6}\cdot\delta_{(132)} - \frac{1}{6}\cdot\delta_{(12)} - \frac{1}{6}\cdot\delta_{(23)} - \frac{1}{6}\cdot\delta_{(13)}$$

on the left, $(0,0,1)$ on the right corresponds to

$$e_E = \frac{2}{3}\cdot\delta_{\text{id}} - \frac{1}{3}\cdot\delta_{(123)} - \frac{1}{3}\cdot\delta_{(132)}$$

on the left.

## 2.7 The case of a commutative $G$, the Fourier transform

We assume throughout this subsection that $G$ is finite and commutative, and that $k$ is algebraically closed with characteristic not dividing $|G|$.

### 2.7.1

We start with the following basic claim:

**Claim 2.75.** *Let $E$ be an irreducible $G$-representation. Then* $\dim E = 1$.

*Proof.* We show first that every element in $G$ acts on $E$ by a scalar. Denote by $\rho : G \to Gl(E)$ the representation. Let $g \in G$. Since $k$ is algebraically closed, there exists an eigenvalue $\lambda \in k$ of $\rho(g)$. Let $E_{g,\lambda} \subset E$ be the $\lambda$-eigenspace of $\rho(g)$. Then for every $h \in G$, since $h$ commutes with $g$, $\rho(h)$ commutes with $\rho(g)$ and hence by linear algebra $\rho(h)$ preserves $E_{g,\lambda}$. Therefore $E_{g,\lambda}$ is a non-zero $G$-subrepresentation of $E$. Hence we must have $E_{g,\lambda} = E$, i.e. $\rho(g)$ is $\lambda \cdot \text{Id}_E$.

As all the elements of $G$ act on $E$ by scalar, every linear subspace of $E$ is invariant under the $G$-action, i.e. is a $G$-subrepresentation, and hence $E$ must be 1-dimensional in order to be irreducible. □

Let us denote by $Ch_k(G)$ the set of characters of $G$, i.e. group homomorphisms $G \to k^\times$. Combining the claim with Exercise 2.3, we obtain:

**Corollary 2.76.** *The family*

$$(k_\chi)_{\chi \in Ch_k(G)}$$

*is an exhaustive family of irreducible $G$-representations (i.e. no two $G$-representations in this family are isomorphic, and every irreducible $G$-representation is isomorphic to one from the family).*

**Remark 2.77.** If $k$ is not algebraically closed, the claim is not true in general. For example, we can look at the group $\mu_3$ of third roots of unity in $\mathbb{C}$ (it is a cyclic group of order 3), and $k = \mathbb{R}$. Then we can consider $\mathbb{C}$ as a 2-dimensional $\mathbb{R}$-vector space, and $\mu_3$ acts on $\mathbb{C}$ by usual multiplication. Then $\mathbb{C}$ viewed as a 2-dimensional $\mu_3$-representation over $\mathbb{R}$ is irreducible.

### 2.7.2

Let us now combine this with Proposition 2.70. We obtain:

**Corollary 2.78** (Fourier transform)**.** *We have an isomorphism of $k$-algebras*

$$\mathcal{F} : k[G] \xrightarrow{\sim} \mathrm{Fun}_k(Ch_k(G)).$$

*Here the algebra structure on the right hand side is pointwise multiplication. The isomorphism sends*

$$\sum_{g \in G} c_g \cdot \delta_g$$

*to the function on $Ch_k(G)$ whose value on $\chi$ is*

$$\sum_{g \in G} c_g \cdot \chi(g).$$

**Exercise 2.10.** *Compare roughly with the "usual" Fourier transform you might have already seen. (maybe insert a few details)*

### 2.7.3

A natural question is to find the inverse of $\mathcal{F}$. For this, it is enough to ask, fixing $\chi \in Ch_k(G)$, what is the element $e_\chi \in k[G]$ for which $\mathcal{F}(e_\chi)(\chi) = 1$ and $\mathcal{F}(e_\chi)(\theta) = 0$ for $\chi \neq \theta \in Ch_k(G)$. We claim that

$$e_\chi = \frac{1}{|G|} \sum_{g \in G} \chi(g)^{-1} \cdot \delta_g.$$

In terms of terminology, this is Fourier inversion, or the formula for the projector on the isotypic component, or the formula for the idempotent, etc. So let $\theta \in Ch_k(G)$. The scalar by which $e_\chi$ acts on $k_\theta$ is

$$\frac{1}{|G|} \sum_{g \in G} \chi(g)^{-1} \theta(g) = \frac{1}{|G|} \sum_{g \in G} (\theta \chi^{-1})(g).$$

We want to show that this is 1 if $\theta = \chi$ and 0 otherwise. Setting $\mu := \theta\chi^{-1}$, we then want to show that

$$\sum_{g \in G} \mu(g)$$

is equal to $|G|$ if $\mu = 1$ and to 0 otherwise. The first case is clear. Let us assume then that $\mu \neq 1$. This means that there exists $g_0 \in G$ such that $\mu(g_0) \neq 1$. Then

$$\sum_{g \in G} \mu(g) = \sum_{g \in G} \mu(g_0 g) = \sum_{g \in G} \mu(g_0)\mu(g) = \mu(g_0) \sum_{g \in G} \mu(g).$$

Therefore

$$(1 - \mu(g_0)) \sum_{g \in G} \mu(g) = 0.$$

Since $1 - \mu(g_0) \neq 0$, we can divide by it and obtain

$$\sum_{g \in G} \mu(g) = 0,$$

as desired.

Notice also that, for $g \in G$, $\delta_g$ acts on $k_\chi$ by the scalar $\chi(g)$, and so

$$\mathcal{F}(\delta_g) = (\chi(g))_{\chi \in Ch_k(G)};$$

applying $\mathcal{F}^{-1}$, we obtain

$$\delta_g = \sum_{\chi \in Ch_k(G)} \chi(g) \cdot e_\chi.$$

### 2.7.4

We can therefore summarize that we have two bases for $k[G]$ - the "geometric" basis $(\delta_g)_{g \in G}$ and the "spectral" basis $(e_\chi)_{\chi \in Ch_k(G)}$[1]. The change-of-basis matrices are

$$e_\chi = \sum_{g \in G} \frac{1}{|G|} \chi(g)^{-1} \cdot \delta_g,$$

$$\delta_g = \sum_{\chi \in Ch_k(G)} \chi(g) \cdot e_\chi.$$

Consider, for $g_0 \in G$, the linear operators

$$\text{Shift}_{g_0} : k[G] \to k[G], \quad \text{Shift}_{g_0}(\delta_g) := \delta_{g \cdot g_0},$$

and

$$\text{Collapse}_{g_0} : k[G] \to k[G], \quad \text{Collapse}_{g_0}(\delta_g) = \begin{cases} \delta_g & \text{if } g = g_0 \\ 0 & \text{if } g \neq g_0 \end{cases}.$$

Then the geometric basis diagnolizes the Collapse operators (and therefore, taking linear combinations, all the operators of multiplying pointwise by a given function on $G$), while the spectral basis diagnolizes the Shift operators. Notice that to define the Collapse operators we didn't need to know the group structure on the set $G$, while to define the Shift operators, we need to know the groups structure[2].

[1] "geometric" since the geometry inside the group $G$ comes into play - isolating one of the $g \in G$ is like pinpointing a locus in $G$ as a geometric object, and "spectral", roughly, because irreducible representations are possible "eigenvalues" or "wave frequencies" - isolating one of the $\chi \in Ch_k(G)$ is like pinpointing a specific wave component of the overall signal (maybe can explain better).

[2] So, again intuitively speaking, the spectral basis should be a basis consisting of elements which are hopefully "almost left unchanged" when you shift them on the group.

### 2.7.5

Let us indicate "the" application for the Fourier transform for finite abelian groups:

**Theorem 2.79** (Dirichlet, 1837)**.** *Let $d \in \mathbb{Z}_{\geq 1}$ and let $a \in \mathbb{Z}$. Assume that $a$ and $d$ are relatively prime. Then there exist infinitely many primes which are congruent to $a$ modulo $d$.*

The difficulty here (so to speak) is that to be congruent to $a$ is an additive condition, while to be prime is a multiplicative condition.

One recasts the problem into the following shape. We consider the group $(\mathbb{Z}/d\mathbb{Z})^\times$ of invertible elements in the ring $\mathbb{Z}/d\mathbb{Z}$ of residues modulo $d$. Given a function $f \in \mathrm{Fun}_{\mathbb{C}}((\mathbb{Z}/d\mathbb{Z})^\times)$, we extend it to a function on $\mathbb{Z}/d\mathbb{Z}$ by setting it to be equal to zero at the remaining elements (by abuse of notation we will denote this extension by $f$ again). We then consider the formal series

$$M_f(s) := \sum_{p \text{ prime}} \frac{f([p]_d)}{p^s}$$

where $[-]_d : \mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$ denotes the canonical projection. Here, let us assume that $s$ is a real parameter.

Let us note that the series $M_f(s)$ converges absolutely when $s > 1$, because it is dominated by a constant (the maximum of absolute values of the values of $f$) times the series $\sum_{n \in \mathbb{Z}_{\geq 1}} n^{-s}$.

Now, denoting by $\delta_a$ the function on $(\mathbb{Z}/d\mathbb{Z})^\times$ which is equal to 1 at $[a]_d$ and to 0 everywhere else, the theorem will clearly follow if we can see that $M_{\delta_a}(s)$ is unbounded as $s$ tends to 1 from the right.

The behavior of $M_f(s)$ as $s$ tends to 1 from the right does not seem to be tractable for $f$ being $\delta_a$. The main point is that it is tractable for $f$ being a character $\chi : (\mathbb{Z}/d\mathbb{Z})^\times \to \mathbb{C}^\times$ (i.e. a group homomorphism). We have the following proposition:

**Proposition 2.80.** *Let $\chi \in Ch_{\mathbb{C}}((\mathbb{Z}/d\mathbb{Z})^\times)$. If $\chi \neq 1$, then $|M_\chi(s)|$ is bounded as $s$ tends to 1 from the right. If $\chi = 1$, then $M_\chi(s)$ is unbounded as $s$ tends to 1 from the right.*

Let us see that this proposition implies the theorem. Indeed, our Fourier theory gives:

$$\delta_a = \frac{1}{|(\mathbb{Z}/d\mathbb{Z})^\times|} \sum_{\chi \in Ch_{\mathbb{C}}((\mathbb{Z}/d\mathbb{Z})^\times)} \chi(a)^{-1} \cdot \chi.$$

In particular, when we write $\delta_a$ as a linear combination of the characters, the trivial character 1 appears with non-zero coefficient. Therefore it is clear from the proposition that $M_{\delta_a}(s)$ is unbounded as $s$ tends to 1 from the right (as

$M_{\delta_a}(s)$ is a finite sum of functions, one of which is unbounded, while all others are bounded).

We now will see how the multiplicative property of the functions $\chi$, being aligned with the multiplicative nature of the primes, allows to transform the study of the series $M_\chi(s)$ into the study of series running over $\mathbb{Z}_{\geq 1}$, so over a set whose structure is much more simpler than that of the structure of the set of primes.

Recall that for $x \in \{z \in \mathbb{C} \mid |z| < 1\})$ we can consider the absolutely converging $-\log(1-x) = \sum_{m\in\mathbb{Z}_{\geq 1}} x^m/m$. We assume that the values of $f$ are bounded in absolute value by 1, and want to replace $a_p(s) := \frac{f([p]_d)}{p^s}$ in the sum defining $M_f(s)$ by $-\log(1-a_p(s))$. We therefore want to consider the series we obtain by considering the absolute values of the differences:

$$\sum_{p \text{ prime}} |-\log(1-a_p(s)) - a_p(s)| = \sum_{p \text{ prime}} \left| \sum_{m\in\mathbb{Z}_{\geq 2}} \frac{a_p(s)^m}{m} \right| \leq \sum_{p \text{ prime}} \sum_{m\in\mathbb{Z}_{\geq 2}} \frac{1}{m \cdot p^{sm}} \leq$$

$$\leq \sum_{p \text{ prime}} \sum_{m\in\mathbb{Z}_{\geq 2}} \frac{1}{p^m} = \sum_{p \text{ prime}} \frac{1}{p^2} \cdot \frac{1}{1-\frac{1}{p}} \leq 2 \sum_{p \text{ prime}} \frac{1}{p^2} \leq 2 \sum_{n\in\mathbb{Z}_{\geq 1}} \frac{1}{n^2}.$$

From this, we deduce that, if the values of $f$ are bounded in absolute value by 1, then $M_f(s)$ is bounded/unbounded as $s$ tends to 1 from the right if and only if

$$\ell_f(s) := \sum_{p \text{ prime}} -\log\left(1 - \frac{f([p]_d)}{p^s}\right)$$

is so. Exponentiating, we need to investigate whether

$$L_f(s) := \prod_{p \text{ prime}} \frac{1}{1-\frac{f([p]_d)}{p^s}}$$

is bounded/unbounded as $s$ tends to 1 from the right. Now, if we take $f$ to be a character $\chi$, then by the unique decomposition of natural numbers into products of primes, we obtain

$$L_\chi(s) = \sum_{n\in\mathbb{Z}_{\geq 1}} \frac{\chi([n]_d)}{n^s}.$$

By further analyzing $L_\chi(s)$, one shows that $L_1(s)$ tends to $+\infty$ as $s$ tends to 1 from the right (this is easy to show, using the divergence of the harmonic series) and this implies that $\ell_1(s)$ is unbounded as $s$ tends to 1 from the right. Also, one shows that if $\chi \neq 0$ then $L_\chi(s)$ tends to a particular value $L_\chi(1)$ in $\mathbb{C}$ as $s$ tends to 1 from the right (this is relatively easy to show). Then it is more difficult to show that $L_\chi(1) \neq 0$ (after all the framework was internalized, this is the main technical point, on which everything depends), and this implies that $\ell_\chi(s)$ is bounded as $s$ tends to 1 from the right.

# 3 Semisimple modules and rings

In this section, we focus on studying a class of rings (called semisimple rings) which have the behavior giving rise to an Artin-Wederburn theorem similar to Proposition 2.70 (the non-commutative Fourier transform being an isomorphism).

Throughout this section, $R$ is a fixed ring (recall, for us a ring is unital (has 1) but is not necessarily commutative). Also, recall that by default an $R$-module for us means a left $R$-module.

## 3.1 Some finiteness properties of modules and rings

### 3.1.1

There are two basic properties of modules, named after two persons.

**Definition 3.1.** Let $M$ be an $R$-module. We say that $M$ is **Noetherian** (resp. **Artinian**) if for every increasing (resp. decreasing) infinite sequence of $R$-submodules

$$M_1 \subset M_2 \subset \ldots \qquad (\text{resp. } \ldots \subset M_2 \subset M_1)$$

in $M$, there exists $r_0 \in \mathbb{Z}_{\geq 1}$ such that $M_r = M_{r_0}$ for $r \geq r_0$.

**Remark 3.2.** We can slightly restate the properties. An $R$-module $M$ is Noetherian if there are no strictly increasing infinite sequences of submodules

$$M_1 \subset M_2 \subset \ldots,$$

where "strictly" means that $M_i \neq M_{i+1}$ for every $i \geq 1$. The restatement in the Artinian case is analogous.

**Example 3.3.** *Suppose that $R$ is a $k$-algebra for some field $k$. Then every $R$-module which is finite-dimensional as a $k$-vector space is both Noetherian and Artinian. Indeed, $R$-submodules are in particular $k$-subvector spaces and the chain conditions are satisfied for subvector-spaces of a finite-dimensional vector space by dimension consideration.*

**Remark 3.4.** Although the conditions of being Noetherian and Artinian look similar, to be Artinian is a much more restrictive condition in practice.

The essence of Noetherianity is not hard to understand concretely:

**Lemma 3.5.** *Let $M$ be an $R$-module. Then $M$ is Noetherian if and only if every $R$-submodule in $M$ is finitely generated.*

*Proof.* Suppose that $M$ is Noetherian and let $N \subset M$ be an $R$-submodule. Suppose, to arrive to a contradiction, that $N$ is not finitely generated. Then given elements $v_1, \ldots, v_r \in N$, we have $Rv_1 + \ldots + Rv_n \neq N$ and therefore we can find an element $v_{r+1} \in N$ such that $Rv_1 + \ldots + Rv_{r+1}$ strictly contains $Rv_1 + \ldots + Rv_r$. Therefore, we can construct a sequence $v_1, v_2, \ldots$ of elements in $N$ such that, denoting $N_r := Rv_1 + \ldots + Rv_r$, we have a strictly increasing sequence $N_1 \subset N_2 \subset \ldots$, contradicting the Noetherian property of $M$.

Suppose conversely that every $R$-submodule of $M$ is finitely generated. Let $M_1 \subset M_2 \subset \ldots$ be an increasing sequence of $R$-submodules in $M$. Denote $N := \cup_{r \geq 1} M_r$. Then $N$ is finitely generated by assumption. Therefore there exist $v_1, \ldots, v_s \in N$ such that $N = Rv_1 + \ldots + Rv_s$. We can find $r_0$ large enough so that $v_1, \ldots, v_s$ all belong to $M_{r_0}$. Then $N$ belongs to $M_{r_0}$ (so $N = M_{r_0}$) and therefore $M_r$ belongs to $M_{r_0}$ for all $r$, so $M_r = M_{r_0}$ for all $r \geq r_0$, showing that $M$ is Noetherian. □

### 3.1.2

**Lemma 3.6.** *Let $M$ be an $R$-module and $N \subset M$ a submodule. Then $M$ is Noetherian (resp. Artinian) if and only if $N$ and $M/N$ are.*

*Proof.* Let us consider the Noetherianity for example. If $M$ is Noetherian then it is clear that $N$ and $M/N$ are Noetherian, since the partially ordered sets of submodules of $N$ and submodules of $M/N$ are embedded in the partially ordered set of submodules of $M$ (the second by sending a submodule of $M/N$ to its inverse image under the canonical projection $M \to M/N$). Let us assume now that $N$ and $M/N$ are Noetherian. Let

$$M_1 \subset M_2 \subset \ldots$$

be an increasing sequence of submodules in $M$. Considering the increasing sequence of submodules

$$M_1 \cap N \subset M_2 \cap N \subset \ldots$$

in $N$, there exists $r_0$ such that $M_r \cap N = M_{r_0} \cap N$ for $r \geq r_0$. Considering the increasing sequence of submodules

$$\frac{M_1 + N}{N} \subset \frac{M_2 + N}{N} \subset \ldots$$

in $M/N$, there exists $r_1$ such that $\frac{M_r + N}{N} = \frac{M_{r_1} + N}{N}$ for $r \geq r_1$. Therefore, setting $r_2 = \max\{r_0, r_1\}$, for $r \geq r_2$ we have $M_r \cap N = M_{r_2} \cap N$ and $M_r + N = M_{r_2} + N$, and from this one deduces $M_r = M_{r_2}$. □

**Remark 3.7.** Of course, the last lemma implies that a finite direct sum of Noetherian (resp. Artinian) modules is Noetherian (resp. Artinian). Indeed, given a direct sum $M_1 \oplus M_2$, we have the submodule $M_1 \oplus 0$ in $M_1 \oplus M_2$ which is isomorphic to $M_1$, and the quotient by it is isomorphic to $M_2$.

#### 3.1.3

We have the following definitions for a ring:

**Definition 3.8.** We say that $R$ is **left Noetherian** (resp. **left Artinian**) if $R$ is Noetherian (resp. Artinian) as a left $R$-module.

**Remark 3.9.** Notice that when we consider $R$ as a left $R$-module, the $R$-submodule of $R$ are left ideals in $R$.

**Example 3.10.** *If $R$ is a finite-dimensional $k$-algebra, then $R$ is left Noetherian and left Artinian. This follows from Example 3.3.*

**Exercise 3.1.** *Show that $R$ is left Noetherian (resp. left Artinian) if and only if all finitely generated (left) $R$-modules are Noetherian (resp. Artinian).*

## 3.2 Simple and semisimple modules

#### 3.2.1

As for representations, we have the notion of irreducibility:

**Definition 3.11.** Let $M$ be an $R$-module. Then $M$ is said to be **simple**, or **irreducible**, if $M \neq 0$ and for every $R$-submodule $N \subset M$ one has either $N = 0$ or $N = M$.

**Exercise 3.2.** *Show that associating to a maximal left ideal $I \subset R$ the $R$-module $R/I$, we obtain a surjective map from the set of maximal left ideals in $R$ to the set of isomorphism classes of simple $R$-modules. Also, show that if $R$ is commutative then this map is also injective, so a bijection.*

Let us record here Schur's lemma again:

**Lemma 3.12** (Schur's lemma)**.** *Let $E$ and $F$ be simple $R$-modules. Then every $R$-module morphism $T : E \to F$ is either $0$ or an isomorphism. In particular, $\mathrm{End}_R(E)$ is a division ring.*

*Proof.* The proof is as before. ☐

#### 3.2.2

We gave the definition of a semisimple representation; that of a semisimple module is the same:

**Definition 3.13.** Let $M$ be an $R$-module. Then $M$ is said to be **semisimple** if for every $R$-submodule $N \subset M$ there exists an $R$-submodule $L \subset M$ such that $M = N \oplus L$.

**Lemma 3.14.**

1. *The direct sum of a finite collection of semisimple modules is semisimple.*

2. *A submodule of a semisimple module is semisimple.*

3. *A quotient module of a semisimple module is semisimple.*

*Proof.*

1. It is enough to consider two semisimple $R$-modules $M_1, M_2$ and to show that $M_1 \oplus M_2$ is semisimple. Let $N \subset M_1 \oplus M_2$ be a submodule. We can think about $M_1$ and $M_2$ as submodules of $M_1 \oplus M_2$ (by inclusions $m_1 \mapsto (m_1, 0)$ and $m_2 \mapsto (0, m_2)$), and we also can think about $M_1$ as a quotient module of $M_1 \oplus M_2$, by using the projection on the first coordinate $p_1 : M_1 \oplus M_2 \to M_1$ (given by $p_1(m_1, m_2) = m_1$). Since $M_1$ and $M_2$ are semisimple, we can find submodules $N_1 \subset M_1$ and $N_2 \subset M_2$ such that $M_1 = p_1(N) \oplus N_1$ and $M_2 = (N \cap M_2) \oplus N_2$. We now claim that $M = N \oplus (N_1 \oplus N_2)$.

   Indeed, let us first see that $N \cap (N_1 \oplus N_2) = 0$. Let $n \in N \cap (N_1 \oplus N_2)$. Then $p_1(n) \in p_1(N) \cap N_1 = 0$ and therefore $p_1(n) = 0$. Therefore $n \in M_2$. So $n \in (N \cap M_2) \cap N_2 = 0$ so $n = 0$.

   Second, we need to check that $M = N + (N_1 \oplus N_2)$. Let $m = (m_1, m_2) \in M$. Then we can write $m_1 = p_1(m)$ as $p_1(n) + n_1$ for some $n \in N$ and $n_1 \in N_1$. This means that $p_1(m - (n + n_1)) = 0$ so $m - (n + n_1) \in M_2$. Therefore we can find $n' \in N \cap M_2$ and $n_2 \in N_2$ such that $m - (n + n_1) = n' + n_2$. So $m = (n + n') + n_1 + n_2$ and $n + n' \in N, n_1 \in N_1, n_2 \in N_2$, showing that $m \in N + (N_1 \oplus N_2)$ as desired.

2. Let $M$ be a semisimple module and $L \subset M$ a submodule. We want to show that $L$ is semisimple. Let $N \subset L$ be a submodule. We can find a submodule $N' \subset M$ such that $M = N \oplus N'$. We now leave to the reader to check that $L = N \oplus (L \cap N')$.

3. Let $M$ be a semisimple module and $L \subset M$ a submodule. We want to show that $M/L$ is semisimple. Let $N \subset M/L$ be a submodule. Let $\widetilde{N} \subset M$ be the preimage of $N$ under the natural projection map $p : M \to M/L$. Since $M$ is semisimple, we can find a submodule $N' \subset M$ such that $M = \widetilde{N} \oplus N'$. We now leave to the reader to check that $M/L = N \oplus p(N')$.

□

**Remark 3.15.** In the above lemma, it is also true that an infinite direct sum of semisimple modules is semisimple. However, this more general statement will use Zorn's lemma, and we leave it as an optional exercise.

### 3.2.3

The relation between semisimplicity and simplicity is as follows:

**Claim 3.16.** *Let $M$ be a finitely generated $R$-module. The following are equivalent:*

1. *$M$ is semisimple.*

2. *$M$ can be written as a finite direct sum of simple modules.*

*Proof.* Suppose that $M$ can be written as a finite direct sum of simple modules. Since a simple module is clearly semisimple, and a finite direct sum of semisimple modules is semisimple, we obtain that $M$ is semisimple.

Suppose now that $M$ is semisimple. Then every submodule of $M$ is isomorphic to a quotient module of $M$, and therefore is finitely generated, as $M$ is. Therefore, $M$ is Noetherian. So every submodule of $M$ is again semisimple and Noetherian.

We now claim that any non-zero submodule of $M$ contains a maximal proper submodule[3]. Indeed, let $N \subset M$ be a non-zero submodule. We can pick any proper submodule $N_1 \subset N$ (for example, 0). If $N_1$ is not a maximal proper submodule of $N$, we can pick another proper submodule $N_2 \subset N$ such that $N_1 \subset N_2$. This will stop eventually because $N$ is Noetherian, and we deduce that $N$ must contain a maximal proper submodule.

If $M$ is zero, the claim is clear ($M$ can be written as an empty direct sum of simple modules). Otherwise, we can find a maximal proper submodule $N_1 \subset M$ and choose a complementary submodule $E_1 \subset M$, so that $M = N_1 \oplus E_1$. Since $N_1$ is a maximal proper submodule of $M$, $E_1$ is simple. If $N_1$ is zero, we are done. Otherwise, we again choose a maximal proper submodule $N_2 \subset N_1$ and a complement $N_1 = N_2 \oplus E_2$, where $E_2$ is again simple. Continuing like this, we obtain simple submodules $E_1, E_2, \ldots$ in $M$, and a strictly increasing sequence of submodules

$$E_1, E_1 \oplus E_2, \ldots$$

Since $M$ is Noetherian, this must stop after finitely many times, meaning that we will find $M = E_1 \oplus \ldots \oplus E_r$, as desired. □

**Remark 3.17.** In the above claim, we can drop the assumption that $M$ is finitely generated, but then also drop the requirement that the direct sum is finite. However, this more general statement will use Zorn's lemma, and we leave it as an optional exercise.

### 3.2.4

Let us prove the following finiteness result:

**Proposition 3.18.** *Suppose that $R$ is left Artinian. Then the set of isomorphism classes of simple $R$-modules is finite.*

*Proof.* We will proceed via proof by contradiction. Assume that we are given an infinite sequence $E_1, E_2, \ldots$ of simple $R$-modules, pairwise non-isomorphic

[3]A submodule is said to be proper if it is not equal to the whole module.

(we want to arrive to contradiction). Choose non-zero $e_i \in E_i$ for each $i$. Let $I_i$ denote the annihilator of $e_i$, i.e.

$$I_i := \{x \in R \mid xe_i = 0\} \subset R.$$

Then $I_i$ is a left ideal in $R$ and we have a natural $R$-module isomorphism $R/I_i \cong E_i$. We claim that, for any $n$, $I_{n+1}$ does not contain $I_1 \cap \ldots \cap I_n$. Having demonstrated that, we obtain a strictly decreasing sequence

$$\ldots \subset I_1 \cap I_2 \cap I_3 \subset I_1 \cap I_2 \subset I_1,$$

contrary to the Artinian property.

So it is left to show that $I_{n+1}$ does not contain $I_1 \cap \ldots \cap I_n$. Suppose to the contrary that $I_1 \cap \ldots \cap I_n \subset I_{n+1}$. Notice that the $R$-module morphism

$$R \to E_1 \oplus \ldots \oplus E_n$$

given by $x \mapsto (xe_1, \ldots, xe_n)$ factors via an injective $R$-modules morphism

$$R/(I_1 \cap \ldots \cap I_n) \hookrightarrow E_1 \oplus \ldots \oplus E_n.$$

Since $E_1 \oplus \ldots \oplus E_n$ is semisimple (as a finite direct sum of simple modules), we can find a complementary module, and using it a surjection

$$E_1 \oplus \ldots \oplus E_n \twoheadrightarrow R/(I_1 \cap \ldots \cap I_n).$$

Notice now that by our assumption we have a surjection

$$R/(I_1 \cap \ldots \cap I_n) \twoheadrightarrow R/I_{n+1} \cong E_{n+1},$$

so overall we obtain a surjection

$$E_1 \oplus \ldots \oplus E_n \twoheadrightarrow E_{n+1}.$$

In particular, this is not zero and hence one of the factors $E_i \to E_{n+1}$, for some $1 \leq i \leq n$, must be not zero. By Schur's lemma, we then have $E_i \cong E_{n+1}$, obtaining a contradiction. $\square$

**Remark 3.19.** We saw above that the set of isomorphism classes of irreducible $G$-representations over a field $k$, when $G$ is finite and the characteristic of $k$ does not divide $|G|$, is finite. The last proposition shows that again, and even allows to drop the condition on $k$ (as $k[G]$ is left Artinian, being a finite-dimensional $k$-algebra).

## 3.3 Semisimple rings

### 3.3.1

**Definition 3.20.** The ring $R$ is said to be **(left) semisimple** if $R$, as a (left) $R$-module, is semisimple.

**Lemma 3.21.** *The ring $R$ is semisimple if and only if all finitely generated $R$-modules are semisimple.*

*Proof.* Since $R$ is a finitely generated $R$-module, one direction is obvious. For the other, assume that $R$ is semisimple. Let $M$ be a finitely generated $R$-module. This means that we can find a surjective morphism of $R$-modules $R^n \to M$ for some $n \in \mathbb{Z}_{\geq 0}$, identifying $M$ with a quotient of $R^n$. We saw that semisimplicity of modules is preserved under finite direct sums and quotients, so $M$ is semisimple as a quotient of a finite direct sum of copies of $R$. $\square$

**Remark 3.22.** In the above lemma, we can drop the finitely generated condition, i.e. if $R$ is semisimple then all $R$-modules are semisimple. But the proof of that requires Zorn's lemma.

**Example 3.23.** *Let $G$ be a finite group and $k$ a field whose characteristic does not divide $|G|$. Then Maschke's theorem says that all $k[G]$-modules which are finite-dimensional as $k$-vector spaces are semisimple, and in particular $k[G]$ itself, viewed as a (left) $k[G]$-module, is semisimple. Hence the $k$-algebra $k[G]$ is semisimple.*

**Example 3.24.** *Let $D$ be a division ring. Then $D$ is semisimple. Indeed, $D$ viewed as a (left) $D$-module is simple, i.e. admits no submodules except $0$ and $D$. From a different perspective, we can see that every $D$-module is semisimple by "completing a basis" as mentioned in §2.6.2.*

**Example 3.25.** *Let $D$ be a division ring and $V$ a finite-dimensional $D$-module. Then $\mathrm{End}_D(V)$ is a semisimple ring. Indeed, let us fix a basis $e_1, \ldots, e_n$ for the $D$-module $V$. For $1 \leq i \leq n$, let us consider the subset $L_i \subset \mathrm{End}_D(V)$ consisting of $T$ for which $T(e_j) = 0$ for $j \neq i$. Then $L_i$ is a left ideal in $\mathrm{End}_D(V)$ (in other words, it is a $\mathrm{End}_D(V)$-submodule of $\mathrm{End}_D(V)$ viewed as a left $\mathrm{End}_D(V)$-module). We have $\mathrm{End}_D(V) = L_1 \oplus \ldots \oplus L_n$ (check is left as an exercise). Finally, each $L_i$ is a simple $\mathrm{End}_D(V)$-module. Indeed, given any two $T_1, T_2 \in L_i$, both non-zero, we can find $S \in \mathrm{End}_D(V)$ such that $S(T_1(e_i)) = T_2(e_i)$ (this is seen as in linear algebra, by completing to a basis). Then $ST_1$ and $T_2$ are equal to $e_i$ and are equal on all other $e_j$ (both being equal to zero on them), and therefore $ST_1 = T_2$. In other words, we explained that the action of $\mathrm{End}_D(V)$ on $L_i \smallsetminus \{0\}$ is transitive, and therefore, it is immediate to see, $L_i$ is a simple $\mathrm{End}_D(V)$-module.*

**Remark 3.26.** Given a division ring $D$, we consider the $D^{op}$-module $V := (D^{op})^n$, and then $\mathrm{End}_{D^{op}}(V) \cong M_{n \times n}(D)$, the ring of $n$ by $n$ matrices over $D$. So a restatement of the previous example is that matrix rings over division algebras are semisimple.

**Exercise 3.3.** *Let $R_1$ and $R_2$ be semisimple rings. Show that $R_1 \times R_2$ is also a semisimple ring.*

#### 3.3.2

**Claim 3.27.** *Suppose that $R$ is semisimple. Then $R$ is left Noetherian and left Artinian.*

*Proof.* Since $R$ is semisimple and $R$ is obviously finitely generated as an $R$-module, Claim 3.16 shows that $R$ can be written as a finite direct sum of simple $R$-modules. Then Remark 3.7 Shows that $R$ is a Noetherian $R$-module and an Artinian $R$-module, as desired. □

**Corollary 3.28.** *Suppose that $R$ is semisimple. Then the set of isomorphism classes of simple $R$-modules is finite.*

*Proof.* This follows from the last claim and Proposition 3.18. □

### 3.4 The Artin-Wedderburn theorem

#### 3.4.1

**Claim 3.29** (Jacobson's density theorem)**.** *Let $M$ be a semisimple $R$-module. Denote $S := \mathrm{End}_R(M)$. Let us be given $T \in \mathrm{End}_S(M)$ and $v_1, \ldots, v_n \in M$. Then there exists $r \in R$ such that $rv_i = T(v_i)$ for all $1 \leq i \leq n$.*

*Proof.* We first deal with the case $n = 1$. Since $M$ is semisimple, we can write $M = Rv_1 \oplus N$ for some $R$-submodule $N \subset M$. The projection on $Rv_1$ along $N$ is an element $P \in S$. Notice that $PTv_1 = TPv_1 = Tv_1$ and therefore $Tv_1 \in Rv_1$, so there exists $r \in R$ such that $Tv_1 = rv_1$, as desired.

Let us now reduce the case of general $n$ to that of $n = 1$. For this, we consider the semisimple $R$-module $M^n$, the vector $(v_1, \ldots, v_n) \in M^n$, and the endomorphism $T^{\oplus n}$ of $M^n$ (given by $T^{\oplus n}(m_1, \ldots, m_n) := (Tm_1, \ldots, Tm_n)$). The ring of endomorphisms of $M^n$ as an $R$-module is naturally identified with the ring of $(n \times n)$-matrices over $S$, and one readily checks that $T^{\oplus n}$ commutes with all the endomorphisms of $M^n$ as an $R$-module. Therefore, by the already established $n = 1$ case, we deduce that there exists $r \in R$ such that $r(v_1, \ldots, v_n) = T^{\oplus n}(v_1, \ldots, v_n)$, so $rv_i = Tv_i$ for all $1 \leq i \leq n$. □

**Corollary 3.30.** *Let $M$ be a semisimple $R$-module. Denote $S := \mathrm{End}_R(M)$. Suppose that $M$ is finitely generated as an $S$-module. Then the natural ring morphism $R \to \mathrm{End}_S(M)$ is surjective.*

#### 3.4.2

For a simple $R$-module $E$, let us denote $D_E := \mathrm{End}_R(E)$ and recall that $D_E$ is a division ring by Schur's lemma. Denote by

$$\mathcal{F}_E : R \to \mathrm{End}_{D_E}(E)$$

the natural ring morphism given by the action of $R$ on $E$.

We can now prove the Artin-Wedderburn theorem:

**Theorem 3.31.** *Assume that $R$ is semisimple. There are finitely many simple $R$-modules, up to isomorphism. Any simple $R$-module $E$ is finite-dimensional*[4] *over $D_E$. Denote by*

$$E_1, \ldots, E_r$$

*an exhaustive family of simple $R$-modules. The ring morphism*

$$\mathcal{F} : R \to \prod_{1 \le i \le r} \mathrm{End}_{D_{E_i}}(E_i)$$

*given by the product $\mathcal{F}_{E_1} \times \ldots \times \mathcal{F}_{E_r}$ is an isomorphism.*

*Proof.* We have already seen that $R$ is left Artinian (Claim 3.27), and in particular the set of isomorphism classes of simple $R$-modules is finite (Corollary 3.28).

Let us show the injectivity of $\mathcal{F}$. An element $r \in R$ which maps under $\mathcal{F}$ to zero, acts by zero on every simple $R$-module. Hence it acts by zero on every $R$-module which can be written as a finite direct sum of simple $R$-modules, and in particular it acts by zero on $R$ itself (as $R$ is semisimple). This last thing means that $rx = 0$ for all $x \in R$, in particular for $x = 1$, which gives $r = 0$.

Let us now show the surjectivity of $\mathcal{F}$. It basically follows from Corollary 3.30 applied to the $R$-module $E_1 \oplus \ldots \oplus E_r$, but let us elaborate on this. We consider the semisimple $R$-module

$$\widetilde{E} := E_1 \oplus \ldots \oplus E_r.$$

We want to understand first $S := \mathrm{End}_R(\widetilde{E})$. As always with endomorphisms of direct sums, we have an isomorphism

$$S = \mathrm{End}_R(\widetilde{E}) \cong \prod_{1 \le i,j \le r} \mathrm{Hom}_R(E_i, E_j),$$

given by sending an element $(\phi_{i,j})_{1 \le i,j \le r}$ on the right to $\phi$ on the left given by

$$\phi(e_1, \ldots, e_r) = (\sum_i \phi_{i,1}(e_i), \ldots, \sum_i \phi_{i,r}(e_i)).$$

However, from Schur's lemma we have $\mathrm{Hom}_R(E_i, E_j) = 0$ for $i \neq j$. Therefore we have

$$S = \mathrm{End}_R(\widetilde{E}) \cong \prod_{1 \le i \le r} \mathrm{Hom}_R(E_i, E_i) = \prod_{1 \le i \le r} D_{E_i}.$$

Now we want to understand $\mathrm{End}_S(\widetilde{E})$. We have the ring morphism

$$\mathrm{End}_{D_{E_1}}(E_1) \times \ldots \times \mathrm{End}_{D_{E_r}}(E_r) \to \mathrm{End}_S(\widetilde{E})$$

[4] Recall that for a module over a division algebra, "finite-dimensional" is just a different term for "finitely-generated".

given by sending $(T_1, \ldots, T_r)$ on the left to $T$ on the right given by $T(e_1, \ldots, e_r) := (T_1(e_1), \ldots, T_r(e_r))$. We claim that this is an isomorphism (only surjectivity is not immediately clear). Indeed, let $T \in \mathrm{End}_S(\widetilde{E})$. Define $T_i \in \mathrm{End}_{D_{E_i}}(E_i)$ by setting $T_i(e)$ (where $e \in E_i$) to be the $i$-th component of $T(0, \ldots, e, \ldots, 0)$ (here the vector inside $T$ has $e$ at the $i$-th place). Consider the element $\Phi_i \in S$ given by identity on the $i$-th component and zero everywhere else. We have $\Phi_i \circ T = T \circ \Phi_i$, and seeing what it concretely gives yields $T(e_1, \ldots, e_r) = (T_1(e_1), \ldots, T_r(e_r))$, as desired.

So, we see that our map $\mathcal{F}$ is exactly identified with the map $R \to \mathrm{End}_S(\widetilde{E})$, and so by Corollary 3.30 we will now that $\mathcal{F}$ is surjective if we can show that $\widetilde{E}$ is finitely generated over $S$. Notice that, clearly, $\widetilde{E} = E_1 \oplus \ldots \oplus E_r$ would be a finitely generated $S$-module if we can show that each $E_i$ is a finitely generated $D_{E_i}$-module.

It is left to show that, given a simple $R$-module $E$, $E$ is finitely generated as an $D_E$-module. Notice that we can interpret

$$E \cong \mathrm{Hom}_R(R, E)$$

(where a morphism $\phi$ on the left maps to $\phi(1)$ on the right), and under this interpretation the $D_E$-module structure on $E$ corresponds to the $D_E$-module structure on $\mathrm{Hom}_R(R, E)$ where given $d \in D_E$ and $\phi \in \mathrm{Hom}_R(R, E)$ the result of acting by $d$ on $\phi$ is given by $d \circ \phi$. We can generalize it to considering any $R$-module $M$, and then considering $\mathrm{Hom}_R(M, E)$ as a $D_E$-module, where the result of acting by $d \in D_E$ on $\phi \in \mathrm{Hom}_R(M, E)$ is set to be $d \circ \phi$. Since $R$ is semisimple, it is isomorphic to a finite direct sum of simple $R$-modules. Therefore, $\mathrm{Hom}_R(M, E)$ is isomorphic, as a $D_E$-module, to a finite direct sum of modules $\mathrm{Hom}_R(F, E)$, where $F$ is a simple $R$-module. It is enough, therefore, to show that such $\mathrm{Hom}_R(F, E)$ is a finitely generated $D_E$-module. If $F$ is not isomorphic to $E$ then $\mathrm{Hom}_R(F, E)$ is zero by Schur's lemma and the claim is clear. If $F$ is isomorphic to $E$, then $\mathrm{Hom}_R(F, E)$ is isomorphic to $\mathrm{Hom}_R(E, E) = D_E$, and this is a finitely generated $D_E$-module (with generator 1). □

**Corollary 3.32** (Also Artin-Wederburn theorem)**.** *A ring is semisimple if and only if it is isomorphic to a finite product of matrix rings over division algebras.*

*Proof.* We just saw that a semisimple ring is isomorphic to a product of matrix rings over division algebras. Conversely, we saw that a matrix ring over a division algebra is semisimple, and that the product of semisimple rings is semisimple. □

**Exercise 3.4.** *One can wonder, writing a semisimple ring $R$ as a product $M_{n_1}(D_1^{op}) \times \ldots \times M_{n_r}(D_r^{op})$ of matrix rings over division algebras, whether $(n_1, D_1), \ldots, (n_r, D_r)$ is a well-defined list, up two replacing $D_i$'s by isomorphic division rings, and order. Show that this is so, as follows. Show that the simple $R$-modules are $D_i^{n_i}$, where $M_{n_i}(D_i^{op})$ acts on this by matrix multiplication, and $M_{n_j}(D_j^{op})$ for $j \neq i$ acts on this by zero. Therefore, understand that the*

*$(n_i, D_i)$'s can be recovered by taking a simple $R$-module $E$, considering the division algebra $D := \mathrm{End}_R(E)$ and the dimension $n$ of $E$ over $D$, giving rise to a pair $(n, D)$. This description does not depend on the chosen decomposition.*

## 3.5 The Jacobson radical

Looking at the "non-commutative Fourier transform" $\mathcal{F}$, and at our establishing its injectivity in certain cases, we see that it makes sense in general to ask about the kernel - i.e. what elements in a ring $R$ act by 0 on all simple $R$-modules - those are the "ghosts" in terms of spectral analysis of $R$, so to speak.

### 3.5.1

**Definition 3.33.** The (left) **Jacobson radical** of $R$ is the subset $J(R) \subset R$ consisting of $x \in R$ such that for every simple $R$-module $E$ one has $xE = 0$.

**Lemma 3.34.**

1. *The Jacobson radical $J(R) \subset R$ is a two-sided ideal in $R$.*
2. *The Jacobson radical $J(R)$ is equal to the intersection of all maximal left ideals in $R$.*
3. *Let $x \in R$. Then $x \in J(R)$ if and only if, for all $y \in R$, $1 - yx$ is left-invertible.*

*Proof.*

1. Easy.

2. First, let us notice a few things about simple $R$-modules. Let $E$ be a simple $R$-module and $0 \neq v \in E$. We have a morphism of $R$-modules $R \to E$ given by $x \mapsto xv$. Since this morphism is non-zero, its image, a non-zero $R$-submodule of the simple $R$-module $E$, must be the whole $E$. So our morphism is surjective, and so it induces an isomorphism of $R$-modules $E \cong R/I$, where $I \subset R$ is the kernel of the morphism, i.e. the annihilator of $v$. Notice that $I$ is a maximal left ideal in $R$. Conversely, if $I \subset R$ is a maximal left ideal in $R$, then $R/I$ is a simple $R$-module and $I$ is the annihilator of $1 + I \in R/I$. Therefore, we see that maximal left ideals in $R$ can be characterized as annihilators of non-zero elements in simple modules. From this, the claim is straightforward.

3. First, recall that an element in $R$ will be left-invertible if and only if the left ideal generated by it is the whole $R$ or, equivalently, the element is not contained in any maximal left ideal. Suppose that $x \in J(R)$. Let $y \in R$. Then $1-yx$ is not contained in any left maximal ideal. Hence $1-yx$ is left-invertible. Conversely, suppose that $x \notin J(R)$. Then there exists a maximal left ideal $I \subset R$ such taht $x \notin I$. Then $Rx + I = R$. Hence, there exist $y \in R, z \in I$ such that $yx + z = 1$. Then $1 - yx \in I$ and so $1 - yx$ is not left-invertible. □

**Remark 3.35.** Recall that if $x \in R$ is nilpotent, then $1 - x$ is invertible (with inverse $1 + x + x^2 + \ldots$). Thus nilpotent elements are "small", or "negligible" elements, such that if you add them to 1 it stays invertible. So the criterion in part 3 of the lemma can be vaguely thought of as saying that the elements of $J(R)$ are "negligible" in some algebraic sense (while the definition of $J(R)$ is to consist of elements which are "negligible" in a spectral sense).

### 3.5.2

Here is the relation between the Jacobson radical and semisimplicity:

**Claim 3.36.** *The following are equivalent:*

1. *$R$ is semisimple.*
2. *$R$ is left Artinian and the Jacobson radical of $R$ is equal to* 0.

*Proof.* If $R$ is semisimple, we have already seen that $R$ is left Artinian. In terms of $\mathcal{F}$ above, we see that $J(R)$ is precisely the kernel of $\mathcal{F}$, and so $J(R) = 0$ by one part (the easier part) of the Artin-Wederburn theorem. Let us therefore now assume that $R$ is left Artinian and that $J(R) = 0$, and show that $R$ is semisimple. Since $R$ is left Artinian, there exists a simple submodule (i.e. minimal left ideal) $I_1 \subset R$. Since $J(R) = 0$ and $J(R)$ is the intersection of all maximal left ideals in $R$, there exists a maximal left ideal $\mathfrak{m} \subset R$ such that $\mathfrak{m}$ does not contain $I_1$. Then $\mathfrak{m} \cap I_1 = 0$ since $I_1$ is simple. Therefore $R = I_1 \oplus \mathfrak{m}$. If $\mathfrak{m} = 0$ then we are done, otherwise we can find a simple submodule $I_2 \subset \mathfrak{m}$ and find a maximal left ideal $\mathfrak{n} \subset R$ such that $R = I_2 \oplus \mathfrak{n}$. Then $\mathfrak{m} = I_2 \oplus (\mathfrak{n} \cap \mathfrak{m})$ and so $R = I_1 \oplus I_2 \oplus (\mathfrak{n} \cap \mathfrak{m})$. We can proceed, showing that if $R = I_1 \oplus \ldots \oplus I_d \oplus J_{d+1}$ with $I_1, \ldots, I_d$ simple, then either $J_{d+1} = 0$ or we can find a simple $I_{d+1} \subset J_{d+1}$ and a $J_{d+2} \subset J_{d+1}$ such that $R = I_1 \oplus \ldots \oplus I_d \oplus I_{d+1} \oplus J_{d+2}$. Continuing, the process must stop because $J_d$ is a decreasing sequence of left ideals and $R$ is left Artinian. Hence we will eventually be able to write $R$ as a finite direct sum of simple $R$-modules, showing that $R$ is semisimple. □

**Example 3.37.** *The examples of $\mathbb{Z}$, or $k[X]$ where $k$ is a field, are of rings whose Jacobson radical is zero, but are not semisimple (not left Artinian, in fact).*

**Exercise 3.5.** *Show that (for any ring $R$) $J(R/J(R)) = 0$.*

**Corollary 3.38.** *Assume that $R$ is left Artinian. Then $R/J(R)$ is semisimple.*

*Proof.* Clearly $R/J(R)$ is left Artinian, and by the exercise we have $J(R/J(R)) = 0$. □

**Corollary 3.39.** *Let $R$ be a left Artinian ring. Let $E_1, \ldots, E_r$ be an exhaustive family of simple $R$-modules. Then the ring morphism*

$$\mathcal{F} : R \to \prod_{1 \leq i \leq r} \mathrm{End}_{D_{E_i}}(E_i)$$

*given by the product $\mathcal{F}_{E_1} \times \ldots \times \mathcal{F}_{E_r}$ is surjective.*

*Proof.* An exercise (deduce it from the isomorphism of the non-commutative Fourier transform for $R/J(R)$, which is a semisimple ring). □

### 3.5.3

**Lemma 3.40** (A vesrion of Nakayama's lemma)**.** *Let $M$ be a finitely generated $R$-module. IF $J(R)M = M$ then $M = 0$.*

*Proof.* Let $v_1, \ldots, v_n \in M$ be a set of generators of $M$ as an $R$-module. We can find elements $x_1, \ldots, x_n \in J(R)$ such that $v_1 = x_1 v_1 + \ldots + x_n v_n$. Then $(1 - x_1)v_1 \in Rv_2 + \ldots + Rv_n$. Since $1 - x_1$ is left-invertible in $R$, we obtain $v_1 \in Rv_2 + \ldots + Rv_n$. Therefore $v_2, \ldots, v_n$ is a set of generators of $M$ as an $R$-module. Continuing, we eventually find that $M = 0$ (check for yourself how it ends in the case $n = 1$). □

**Remark 3.41.** We have the following intuition. Suppose that $R$ is commutative. Then we have a "space" $X$ (formally, the spectrum of $R$), the ring of functions on which is identified with $R$. For an ideal $J \subset R$, we have the "subspace" $X_J \subset X$ (formally, closed subscheme), consisting of the points on which functions from $J$ are zero. Corresponding to a finitely-generated $R$-module $M$ is a "bundle" $\mathcal{M}$ on $X$ (formally, a coherent sheaf on $X$), whose set of sections over $X$ is identified with $M$. Then $M/JM$ has an interpretation as the set of sections of $\mathcal{M}$ over $X_J$. Nakayama's lemma then says that for $J := J(R)$, if the set of sections of $\mathcal{M}$ over $X_J$ consists only of the zero section, then $\mathcal{M} = 0$. This means that $X_J$ must be "almost" equal to $X$. Dually, that $J$ must be "very small" (close to being 0).

### 3.5.4

**Claim 3.42.**

1. *Every nilpotent left ideal in $R$ is contained in $J(R)$.*
2. *Suppose that $R$ is left Artinian. Then $J(R)$ is nilpotent.*

*Proof.*

1. Let $I \subset R$ be a nilpotent left ideal in $R$. So $I^n = 0$ for some $n \in \mathbb{Z}_{\geq 1}$. Let $E$ be a simple $R$-module. If $IE \neq 0$, then since $E$ is simple we obtain $IE = E$, and so inductively if $I^r E = E$ then $I^{r+1}E = II^r E = IE = E$. In particular, $0 = I^n E = E$ - a contradiction. Therefore $IE = 0$. In other words, the elements of $I$ annihilate all simple $R$-modules, and therefore $I \subset J(R)$.

2. Let us consider the ideals $J(R)^n$, for $n \in \mathbb{Z}_{\geq 1}$. These form a decreasing sequence, and therefore, since $R$ is left Artinian, there exists $n_0 \in \mathbb{Z}_{\geq 1}$ such that $J(R)^n = J(R)^{n_0}$ for all $n \geq n_0$. Let us denote $I := J(R)^{n_0}$. The proof will be complete if we show that $I = 0$. Notice that for any $n \in \mathbb{Z}_{\geq 1}$ we have $J(R)^n I = J(R)^n J(R)^{n_0} = J(R)^{n_0+n} = J(R)^{n_0} = I$ (in particular, $J(R)I = I$ and $I \cdot I = I$). Therefore, if $I$ is finitely generated then by Nakayama's lemma we obtain $I = 0$, as desired. We now want to proceed in general, not knowing apriori that $I$ is finitely generated. Let us

assume, to reach a contradiction, that $I \neq 0$. Consider the family $\mathcal{I}$ of left ideals $J$ in $R$, for which $IJ \neq 0$. The family $\mathcal{J}$ is not empty (as it contains $R$ itself) and therefore, since $R$ is left Artinian, the family $\mathcal{J}$ contains minimal elements, denote by $J_0$ one such. Notice that $IJ_0$ also lies in $\mathcal{J}$ (since $I(IJ_0) = (I \cdot I)J_0 = IJ_0 \neq 0$), and therefore by the minimality of $J_0$ we obtain $IJ_0 = J_0$. Then we also have $J(R)J_0 = J(R)(IJ_0) = (J(R)I)J_0 = IJ_0 = J_0$. Therefore, again, if we will see that $J_0$ is a finitely generated left ideal, then by Nakayama's lemma we will obtain $J_0 = 0$ - a contradiction. It is therefore left to see that $J_0$ is a finitely generated left ideal. Since $IJ_0 \neq 0$, there exists $x \in J_0$ such that $Ix \neq 0$ and therefore $Rx \in \mathcal{J}$. By the minimality of $J_0$, we have $Rx = J_0$. So $J_0$ is a finitely generated left ideal.

□

**Corollary 3.43.** *Suppose that $R$ is commutative. Then $J(R)$ contains all nilpotent elements. If $R$ is also Artinian, then $J(R)$ consists precisely of all nilpotent elements.*

**Remark 3.44.** In fact, if $R$ is commutative, the ideal of nilpotent elements can be shown to be equal to the intersection of all prime ideals, while the Jacobson radical, as we saw, is equal to the intersection of all maximal ideals. The above corollary states that if $R$ is Artinian, those coincide. But, in fact, those coincide in much more cases. For example, in the case when $R$ is finitely generated over a field (the case most important to algebraic geometry). See the notion of a Jacobson commutative ring as well as Hilbert's Nullstellensatz for further information.

**Remark 3.45.** We could have shown that over a semisimple ring, every module is semisimple (we showed this for finitely generated modules, the general case requires Zorn's lemma). One also shows easily that a semisimple module is Artinian if and only if it is Noetherian. Equipped with this, we can show now that a left Artinian ring is left Noetherian (!). Indeed, let $R$ be a left Artinian ring. Then we saw that $J(R)^n = 0$ for some $n \in \mathbb{Z}_{\geq 1}$. To show that $R$ is a Noetherian $R$-module, it suffices to show that $R/J(R), J(R)/J(R)^2, \ldots, J(R)^{n-1}/J(R)^n$ are Noetherian $R$-modules. Notice that $J(R)$ annihilates all these modules, and therefore we can consider them as $R/J(R)$-modules, and it is clear that they are Noetherian as $R$-modules if and only if they are Noetherian as $R/J(R)$-modules (as the submodule partially ordered set does not change). Since $R/J(R)$ is semisimple, it is enough to show that they are Artinian $R/J(R)$-modules, which is the same as showing that they are Artinian $R$-modules. But this is clear, since they are all subquotients of the Artianian $R$-module $R$ (a subquotient is a quotient module of a submodule).

### 3.5.5

Suppose that $R$ is a finite-dimensional algebra over a field $k$. Given $x \in R$, we can consider the $k$-linear transformation $m_x : R \to R$ given by $m_x(y) := xy$.

Let us denote by $tr_R(x)$ the trace of $m_x$. Then $(x,y) \mapsto tr_R(xy)$ is a symmetric $k$-bilinear form on $R$. Recall that the **radical**, or **kernel** of this bilinear form is then by definition

$$\{x \in R \mid tr_R(xy) = 0 \ \forall y \in R\}.$$

**Claim 3.46.** *Suppose that $R$ is a finite-dimensional algebra over a field $k$. Then $J(R)$ is contained in the radical of the symmetric $k$-bilinear form $(x,y) \mapsto tr_R(xy)$. In particular, if this $k$-bilinear form is non-degenerate then $R$ is semisimple.*

*Proof.* Let $x \in J(R)$. Then for any $y \in R$ we also have $xy \in J(R)$ and by Claim 3.42 we have that $xy$ is nilpotent. Therefore $m_{xy}$ is nilpotent and therefore $tr_R(xy) = tr(m_{xy}) = 0$, so that $x$ lies in the radical of our symmetric $k$-bilinear form. □

**Remark 3.47.** The converse of the claim is not true in general. Namely, we can find a field $k$ and a finite-dimensional $k$-algebra $R$, such that $R$ is semisimple but the symmetric $k$-bilinear form $(x,y) \mapsto tr_R(xy)$ on $R$ is not non-degenerate. Namely, we take $k$ to be a field of positive characteristic $p$, and $\alpha \in k$ be an element which has no $p$-th root in $k$. We then take $R := k(\sqrt[p]{\alpha})$ (the field extension of $k$ of degree $p$). Then one can show that $tr_R = 0$, but $R$ is of course semisimple, being a field.

### 3.5.6

Let us reprove (once more) Maschke's theorem using Claim 3.46.

*Fourth proof of Maschke's theorem.* Recall that $G$ is a finite group and $k$ a field whose characteristic does not divide $|G|$. We want to show that finite-dimensional $G$-representations over $k$ are semisimple. As we explained before, we can think of $G$-representations over $k$ as $k[G]$-modules. Certainly finite-dimensional $G$-representations over $k$ will be finitely generated $k[G]$-modules. Hence, we would like to establish that $k[G]$ is a semisimple $k$-algebra. By Claim 3.46 it is enough to show that the symmetric $k$-bilinear form $(x,y) \mapsto tr_{k[G]}(xy)$ on $k[G]$ is non-degenerate. We have the $k$-basis $(\delta_g)_{g \in G}$ for $k[G]$. It is easy to see that $tr_{k[G]}(\delta_g)$ is equal to 0 if $g \neq 1$ and to $|G|$ if $g = 1$. Therefore given $0 \neq d = \sum_{g \in G} c_g \cdot \delta_g \in k[G]$, and choosing $h \in G$ such that $c_h \neq 1$, we have $tr_{k[G]}(d \cdot \delta_{h^{-1}}) = |G| \cdot c_h \neq 0$ (the inequality is since $|G|$ is not equal to 0 in $k$, by assumption). This shows that our symmetric bilinear form is non-degenerate, as desired. □

# 4 Tensor products

This is a short section introducing tensor products.

## 4.1 Basic definition

### 4.1.1

Recall that given a ring $R$, we can talk about left $R$-modules (which we by default call $R$-modules) and right $R$-modules. A right $R$-module is an abelian group $M$ equipped with a biadditive action map $M \times R \to M$ (as usual, we write the image of $(m, r)$ simply as $mr$), such that $m1 = m$ for all $m \in M$ and $m(r_1 r_2) = (mr_1)r_2$ for all $m \in M$ and $r_1, r_2 \in R$. Equivalently, this is the data of a morphism of rings $R^{op} \to \mathrm{End}(M)$. So, to give a right $R$-module is the same as to give a left $R^{op}$-module.

### 4.1.2

Let $R$ be a ring, let $M$ be a left $R$-module and let $N$ be a right $R$-module. Let $A$ be an abelian group. A biadditive map $\Phi : N \times M \to A$ is said to be **balanced** if

$$\Phi(nr, m) = \Phi(n, rm) \quad \forall n \in N, m \in M, r \in R.$$

### 4.1.3

Let $R$ be a ring, let $M$ be a left $R$-module and let $N$ be a right $R$-module. We will now describe an abelian group $N \underset{R}{\otimes} M$, equipped with a balanced biadditive map $\Phi^{univ} : N \times M \to N \underset{R}{\otimes} M$, such that the following (called the **universal property of the tensor product**) holds: Given an abelian group $A$ and a balanced map $\Phi : N \times M \to A$, there exists a unique abelian group morphism $T : N \underset{R}{\otimes} M \to A$ such that $\Phi = T \circ \Phi^{univ}$. In a diagram:

$$\begin{array}{ccc} N \times M & \xrightarrow{\Phi^{univ}} & N \underset{R}{\otimes} M \\ & \searrow^{\Phi} & \vdots\ \exists! \\ & & A \end{array}$$

One calls $N \underset{R}{\otimes} M$ the **tensor product** of $N$ and $M$ over $R$. One should think of $\Phi^{univ}$ as part of the structure, but as a matter of usual abbreviation, one doesn't remember the piece of notation $\Phi^{univ}$, and one simply writes, given $n \in N$ and $m \in M$, $n \otimes m$ for $\Phi^{univ}(n, m)$ (it is an element of $N \underset{R}{\otimes} M$).

In fact, the above property characterizes $N \underset{R}{\otimes} M$ in the following sense (it is a basic sense to internalize, that of characterizing an object by an universal property). Suppose that we are given two abelian groups $C_1$ and $C_2$ (here $C$ stands for "candidate") together with balanced maps $\Phi_1^{univ} : N \times M \to C_1$ and $\Phi_2^{univ} : N \times M \to C_2$, both satisfying the above property, i.e. for any abelian group $A$ and a balanced map $\Phi : N \times M \to A$, given $i \in \{1, 2\}$ there exists a unique abelian group morphism $T_i : C_i \to A$ such that $\Phi = T_i \circ \Phi_i^{univ}$. Then we claim that there exists a unique abelian group isomorphism $\iota_{12} : C_1 \to C_2$ such that $\Phi_2^{univ} = \iota_{12} \circ \Phi_1^{univ}$. Indeed, there exists a unique abelian group morphism $\iota_{12}$ as desired, by the property of $C_1$. We want to see that it is an isomorphism. By the property of $C_2$, there exists a unique abelian group morphism $\iota_{21} : C_2 \to C_1$ such that $\Phi_1^{univ} = \iota_{21} \circ \Phi_2^{univ}$. If we consider the composition $\iota_{21} \circ \iota_{12} : C_1 \to C_1$, it satisfies $\Phi_1^{univ} = \iota_{21} \circ \iota_{21} \circ \Phi_1^{univ}$, but also the identity $\mathrm{Id}_{C_1}$ satisfies $\Phi_1^{univ} = \mathrm{Id}_{C_1} \circ \Phi_1^{univ}$ and therefore by the uniqueness part of the property of $C_1$ we must have $\mathrm{Id}_{C_1} = \iota_{21} \circ \iota_{12}$. Analogously, we find $\mathrm{Id}_{C_2} = \iota_{12} \circ \iota_{21}$. Therefore $\iota_{12}$ and $\iota_{21}$ are mutually inverse, so isomorphisms.

In the above sense, we have already defined the abelian group $N \underset{R}{\otimes} M$ together with a balanced biadditive map $N \times M \to N \underset{R}{\otimes} M$, and we only need to check that a "model" for it exists. One then strives to work with $N \underset{R}{\otimes} M$ only using the above universal property, without recourse to a specific model. This is a very important ideology, but it might take time to get used to it, and we will not stress it in this course.

#### 4.1.4

Let $R$ be a ring, let $M$ be a left $R$-module and let $N$ be a right $R$-module. Let us now construct a "model" for $N \underset{R}{\otimes} M$ as above. We consider first the free abelian group $\mathbb{Z}[N \times M]$ with basis corresponding to the set $N \times M$ (let us write $\delta_{(n,m)}$ for the element of the basis corresponding to an element $(n, m) \in N \times M$). We then define $N \underset{R}{\otimes} M$ to be the quotient of $\mathbb{Z}[N \times M]$ by the abelian subgroup generated by all the following elements:

$$\delta_{(n_1+n_2,m)} - \delta_{(n_1,m)} - \delta_{(n_2,m)}$$

$$\delta_{(n,m_1+m_2)} - \delta_{(n,m_1)} - \delta_{(n,m_2)}$$

$$\delta_{(nr,m)} - \delta_{(n,rm)}.$$

We define the map $\Phi^{univ} : N \times M \to N \underset{R}{\otimes} M$ by simply sending $(n, m)$ to the image of $\delta_{(n,m)} \in \mathbb{Z}[N \times M]$ in the quotient group $N \underset{R}{\otimes} M$. It is an exercise now to check that the desired universal property is satisfied.

## 4.2 Basic cases

#### 4.2.1

Let $R$ be a ring, let $M$ be a left $R$-module and let $N$ be a right $R$-module. Suppose that we are given an $R$-basis $(e_i)_{i \in I}$ for $M$, i.e. every element of $M$ can be written as $\sum_{i \in I} r_i \cdot e_i$ for a unique vector $(r_i)_{i \in I}$ of elements in $R$, all of which, except finitely many, are zero. We have a balanced map $N \times M \to \bigoplus_{i \in I} N \cdot$ "$e_i$" (where the "$e_i$" are just placeholders) given by $(n, \sum_{i \in I} r_i e_i) \mapsto \sum_{i \in I} n r_i \cdot$ "$e_i$". We claim that this furnishes the tensor product $N \underset{R}{\otimes} M$. In other words, one needs to check that given a balanced map $\Phi : N \times M \to A$, there exists a unique abelian group morphism $\phi : \bigoplus_{i\in I} N \cdot \text{“}e_i\text{”} \to A$ such that $\Phi(n, \sum_{i\in I} r_i e_i) = \phi(\sum_{i\in I} n r_i \cdot \text{“}e_i\text{”})$. Uniqueness is clear, as we see that we must have $\phi(n \cdot \text{“}e_i\text{”}) = \Phi(n, e_i)$. Existence is also easily checked - we define $\phi(\sum_{i\in I} n_i \cdot \text{“}e_i\text{”}) := \sum_{i\in I} \Phi(n_i, e_i)$ and do the routine check that everything is as desired.

### 4.2.2

Let $R$ be a ring, let $M$ be a left $R$-module and let $N$ be a right $R$-module. Let $S$ be another ring, and suppose that $N$ is also a left $S$-module, such that $(sn)r = s(nr)$ (in such a situation, one says that $N$ is an $(S, R)$**-bimodule**). We then give $N \underset{R}{\otimes} M$ the structure of a left $S$-module as follows. Let $s \in S$, and consider the map $N \times M \to N \underset{R}{\otimes} M$ given by $(n, m) \mapsto (sn) \otimes m$. Then one checks that this is a balanced map, and therfore we obtain a unique abelian group morphism $N \underset{R}{\otimes} M \to N \underset{R}{\otimes} M$ satisfying $n \otimes m \mapsto (sn) \otimes m$. We let this be the action of $s$ on $N \underset{R}{\otimes} M$, and check (using the universal property etc.) that this gives $N \underset{R}{\otimes} M$ the structure of a left $S$-module.

Similarly, if $M$ is a right $S$-module such that $(rm)s = r(ms)$, then $N \underset{R}{\otimes} M$ gets equipped with a structure of a right $S$-module.

### 4.2.3

Let $R, S$ be rings and let $\iota : R \to S$ be a ring morphism. Let $M$ be a left $R$-module. A very important construction is **base change** - a left $S$-module $S \underset{R}{\otimes} M$. Here , in order to form the tensor product, $S$ is considered as a right $R$-module by $s * r := s\iota(r)$ (one usually abuses notation and denotes this simply as $sr$). Since $S$ is also a left $S$-module in the standard way, and we have the commutation $(s's)\iota(r) = s'(s\iota(r))$ by the associativity of $S$, by the construction of §4.2.2 we obtain a left $S$-module structure on $S \underset{R}{\otimes} M$.

**Example 4.1.** *Suppose that $(e_i)_{i\in I}$ is a basis of $M$ as an $R$-module. Then $(1 \otimes e_i)_{i\in I}$ is a basis of $S \underset{R}{\otimes} M$ as an $S$-module. This follows from §4.2.1.*

**Example 4.2.** *The first example of base change that a student usually sees is complexification. Given an $\mathbb{R}$-vector space $V$, we can form the $\mathbb{C}$-vector space $\mathbb{C} \underset{\mathbb{R}}{\otimes} V$. If $(e_i)_{i\in I}$ is a $\mathbb{R}$-basis for $V$, then $(1 \otimes e_i)_{i\in I}$ is a $\mathbb{C}$-bsasis for $\mathbb{C} \underset{\mathbb{R}}{\otimes} V$. So we just allowed formally to multiply basis elements by complex numbers, but in a way that is a-priori canonical, not depending on a choice of a basis.*

**Example 4.3.** *Of course the previous example generalizes as follows. If $k \subset K$ is a field extension, then given a $k$-vector space $V$ we can consider the $K$-vector space $K \underset{k}{\otimes} V$. If $(e_i)_{i\in I}$ is a $k$-basis for $V$, then $(1 \otimes e_i)_{i\in I}$ is a $K$-basis for $K \underset{k}{\otimes} V$.*

#### 4.2.4

Let $R$ be a commutative ring. Then every left $R$-module $M$ we can consider as a right $R$-module by defining $mr := rm$, and vice versa. We will think via this identification in what follows. Let $M$ and $N$ be $R$-modules. Then $N \underset{R}{\otimes} M$, is naturally an $R$-module, as follows. We use the construction of §4.2.2, with $S := R$. Then $N$ is a left $R$-module and $(r_1 n) r_2) = r_1(n r_2)$, and therefore we obtain a left $R$-module structure on $N \underset{R}{\otimes} M$. Similarly we obtain a right $R$-module structure on $N \underset{R}{\otimes} M$ by using the right $R$-module structure of $M$, but these are identified: $r(n \otimes m) = (rn) \otimes m = (nr) \otimes m = n \otimes rm = n \otimes mr = (n \otimes m)r$.

Notice that given an $R$-module $U$, an $R$-bilinear map $N \times M \to U$ is in particular balanced. We can formulate the following universal property (exercise). Given an $R$-module $U$ and a $R$-bilinear map $\Phi : N \times M \to U$, there exists a unique $R$-module morphism $T : N \underset{R}{\otimes} M \to U$ such that $T(n \otimes m) = \Phi(n, m)$ for all $(n, m) \in N \times M$.

In other words, for modules over a commutative ring $R$, we can characterize the tensor product of $R$-modules working exclusively with $R$-modules, not mentioning abelian groups.

#### 4.2.5

A very common case is of vector spaces over a field $k$. Since $k$ is a commutative ring, the previous construction gives, given $k$-vector spaces $V$ and $W$, a $k$-vector space $V \underset{k}{\otimes} W$. To explain $V \underset{k}{\otimes} W$ more concretely, one has the following property. Let $(e_i)_{i \in I}$ be a $k$-basis for $V$ and $(f_j)_{j \in J}$ a $k$-basis for $W$. Then $(e_i \otimes f_j)_{(i,j) \in I \times J}$ is a $k$-basis for $V \underset{k}{\otimes} W$. To show this, construct a vector space $V \underset{k}{\otimes}' W$ with basis "$e_i \otimes f_j$" (paramterized by $I \times J$), construct the map $V \otimes W \to V \underset{k}{\otimes}' W$ as the unique bilinear map sending $(e_i, f_j)$ to $e_i$"$\otimes$"$f_j$, and check that it satisfies the universal property (it is an exercise to understand the details).

#### 4.2.6

(maybe add tensor product of rings/algebras)

## 4.3 Basic properties

#### 4.3.1

Let $k$ be a field, and let $V$ and $W$ be $k$-vector spaces. We have a $k$-linear map $V^* \underset{k}{\otimes} W \to \mathrm{Hom}_k(V, W)$, corresponding to the $k$-bilinear map $V^* \times W \to \mathrm{Hom}_k(V, W)$ defined by sending $(\alpha, w) \in V^* \times W$ to the endomorphism given by sending $v$ to $\alpha(v) \cdot w$.

**Claim 4.4.** *Suppose that $V$ is finite-dimensional. Then the above $k$-linear map $V^* \underset{k}{\otimes} W \to \mathrm{Hom}_k(V, W)$ is an isomorphism.*

*Proof.* Choose a basis $(e_i)_{i\in I}$ for $V$ and a basis $(f_j)_{j\in J}$ for $W$. Let $(e_i^*)_{i\in I}$ be the basis of $V^*$ which is dual to $(e_i)_{i\in I}$. Then $(e_i^* \otimes f_j)_{(i,j)\in I\times J}$ is a basis for $V^* \underset{k}{\otimes} W$. We want to check that the image of this basis under our map is a basis for $\mathrm{Hom}_k(V, W)$. This image is the family $(T_{ij})_{(i,j)\in I\times J}$ where $T_{ij}$ is the $k$-linear map sending $e_i$ to $f_j$ and $e_{i'}$ to 0, for $i' \neq i$. By linear algebra, this is indeed a basis for $\mathrm{Hom}_k(V, W)$. □

## 4.4 Tensor product of representaitons

Let $G$ be a group and $k$ a field. We will work with $G$-representations over $k$.

### 4.4.1

Given $G$-representations $V$ and $W$ over $k$, let us define a $G$-representation $V \underset{k}{\otimes} W$. As a vector space it is $V \underset{k}{\otimes} W$, and we need to define the $G$-action. Let $g \in G$. We have a $k$-bilinear map $V \times W \to V \underset{k}{\otimes} W$ given by $(v, w) \mapsto (gv) \otimes (gw)$. We obtain the corresponding $k$-linear map $V \underset{k}{\otimes} W \to V \underset{k}{\otimes} W$, and let this be the action of $g$. One checks that in this way we obtain a $G$-action.

### 4.4.2

Let us also introduce the **dual**, or **contragradient**, construction. Let $V$ be a $G$-representation. We define a $G$-representation $V^*$. As a vector space it is $V^*$, and we need to define the $G$-action. Let $g \in G$ and let $\alpha \in V^*$. We define $g\alpha$ to be the functional given by $v \mapsto \alpha(g^{-1}v)$.

### 4.4.3

Let $V$ and $W$ be $G$-representations. We consider the above $k$-linear map $V^* \underset{k}{\otimes} W \to \mathrm{Hom}_k(V, W)$. Notice that all ingredients have the structure of a $G$-representation (we gave the dual space, the tensor product and the Hom-space induced $G$-representation structures). One checks immediately that this $k$-linear map is a $G$-morphism. In particular:

**Corollary 4.5.** *Suppose that $V$ is finite-dimensional. Then the $G$-representations $V^* \underset{k}{\otimes} W$ and $\mathrm{Hom}_k(V, W)$ are naturally isomorphic.*

# 5 Character theory

Throughout this section, we fix a finite group $G$ and a ground field $k$ whose characteristic does not divide $|G|$. Thus, all vector spaces, algebras, $G$-representations etc. are over $k$.

## 5.1 Definition and orthogonality

### 5.1.1

**Definition 5.1.** Let $V$ be a finite-dimensional $G$-representation. The **character** $\mathrm{ch}_V \in \mathrm{Fun}_k(G)$ is defined by:

$$\mathrm{ch}_V(g) := \mathrm{Tr}\left(g \curvearrowright V\right).$$

**Definition 5.2.** The space of **class functions** on $G$ is the subspace

$$\mathrm{Fun}_k(G)^{cl} \subset \mathrm{Fun}_k(G)$$

consisting of the functions $f$ which satisfy $f(hgh^{-1}) = f(g)$ for all $g, h \in G$ (equivalently, $f(hg) = f(gh)$ for all $g, h \in G$).

**Lemma 5.3.** *Let $V$ be a finite-dimensional $G$-representation. Then $\mathrm{ch}_V \in \mathrm{Fun}_k(G)^{cl}$.*

*Proof.* This is clear, as $\mathrm{Tr}(TST^{-1}) = \mathrm{Tr}(S)$ for linear endomorphisms $T, S$ of a finite-dimensional vector space. □

**Example 5.4.** *Let $\chi : G \to k^\times$ be a character, i.e. a group homomorphism. Then $\mathrm{ch}_{k_\chi} = \chi$. Notice that there might be a slight confusion in terminology due to these two uses of the word "character", which are related.*

**Example 5.5.** *Let $X$ be a finite $G$-set. Recall the $G$-representation $k[X]$. Then $\mathrm{ch}_{k[X]}(g)$ is equal to the number of fixed points of the auto-bijection $g$ gives on $X$.*

**Exercise 5.1.** *Assume that $k$ has characteristic $0$ and let $V$ be a finite-dimensional $k$-vector space. Let $T \in End(V)$. Define a generating series*

$$A(x) := \sum_{k \geq 0} Tr(T^k)x^k.$$

*Denote $n := \dim(V)$ and denote by $p_T$ the characteristic polynomial of $T$. Show that*

$$exp \int \frac{A(x) - n}{x} = \frac{1}{x^n p_T(x^{-1})}$$

*or equivalently*

$$\frac{d}{dx} log \frac{1}{x^n p_T(x^{-1})} = \frac{A(x) - n}{x}.$$

*Thus, knowing the traces of all powers is equivalent to knowing the characteristic polynomial.*

### 5.1.2

We will now give several elementary calculations regarding the character, which will help in establishing the orthogonality relations.

**Definition 5.6.** For functions $f_1, f_2 \in \mathrm{Fun}_k(G)$ let us define $f_1 \cdot f_2 \in \mathrm{Fun}_k(G)$ by

$$(f_1 \cdot f_2)(g) := f_1(g) \cdot f_2(g).$$

**Lemma 5.7.** *Let $V$ and $W$ be finite-dimensional $G$-representations. Then*

$$\mathrm{ch}_{V \underset{k}{\otimes} W} = \mathrm{ch}_V \cdot \mathrm{ch}_W.$$

*Proof.* Let $(e_i)_{i \in I}$ be a $k$-basis for $V$ and $(f_j)_{j \in J}$ a $k$-basis for $W$. Then $(e_i \otimes f_j)_{(i,j) \in I \times J}$ is a $k$-basis for $V \underset{k}{\otimes} W$. Let $g \in G$. Write $ge_i = \sum_{i'} c_{i,i'} e_{i'}$ and $gf_j = \sum_{j'} d_{j,j'} f_{j'}$. Then

$$g(e_i \otimes f_j) = (ge_i) \otimes (gf_j) = (\sum_{i'} c_{i,i'} e_{i'}) \otimes (\sum_{j'} d_{j,j'} f_{j'}) = \sum_{i',j'} c_{i,i'} d_{j,j'} \cdot e_{i'} \otimes f_{j'}$$

and therefore

$$\mathrm{Tr}(g \curvearrowright V \underset{k}{\otimes} W) = \sum_{i,j} c_{i,i} \cdot d_{j,j} = (\sum_i c_{i,i}) \cdot (\sum_j d_{j,j}) = \mathrm{Tr}(g \curvearrowright V) \cdot \mathrm{Tr}(g \curvearrowright W),$$

as desired. □

**Definition 5.8.** For a function $f \in \mathrm{Fun}_k(G)$ let us define $f^* \in \mathrm{Fun}_k(G)$ by

$$f^*(g) := f(g^{-1}).$$

**Lemma 5.9.** *Let $V$ be a finite-dimensional $G$-representation. Then* $\mathrm{ch}_{V^*} = \mathrm{ch}_V^*$.

*Proof.* Let $(e_i)_{i \in I}$ be a $k$-basis for $V$. Let $(e_i^*)_{i \in I}$ be the corresponding dual $k$-basis for $V^*$. Let $g \in G$. Write $g^{-1} e_i = \sum_{i'} c_{i,i'} e_{i'}$. Then $\mathrm{Tr}(g^{-1} \curvearrowright V) = \sum_i c_{i,i}$. On the other hand $ge_i^*$ sends to $e_{i'}$ to $e_i^*(g^{-1} e_{i'}) = c_{i',i}$. Therefore we also have $\mathrm{Tr}(g \curvearrowright V^*) = \sum_i c_{i,i}$. □

**Corollary 5.10.** *Let $V$ and $W$ be finite-dimensional $G$-representations. Then*

$$\mathrm{ch}_{\mathrm{Hom}(V,W)} = \mathrm{ch}_V^* \cdot \mathrm{ch}_W.$$

*Proof.* This follows from the above lemmas and the isomorphism of $G$-representations $V^* \underset{k}{\otimes} W \cong \mathrm{Hom}_k(V, W)$. □

**Definition 5.11.** For a function $f \in \mathrm{Fun}_k(G)$ let us define $Av(f) \in k$ by

$$Av(f) := \frac{1}{|G|} \sum_{g \in G} f(g).$$

**Lemma 5.12.** *Let $V$ be a finite-dimensional $G$-representation. We have*

$$Av(\mathrm{ch}_V) = \dim V^G.$$

*Proof.* Recall the averaging operator $Av_V^G : V \to V$ given by

$$Av_V^G(v) := \frac{1}{|G|} \sum_{g \in G} gv.$$

We saw before that $Av_V^G$ is a projection operator on the subspace $V^G$. Therefore $\mathrm{Tr}(Av_V^G \curvearrowright V) = \dim V^G$. But

$$\mathrm{Tr}(Av_V^G) = \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}(g \curvearrowright V) = \frac{1}{|G|} \sum_{g \in G} \mathrm{ch}_V(g).$$

□

**Definition 5.13.** Let us define a symmetric bilinear form $\langle -, - \rangle$ on $\mathrm{Fun}_k(G)$ by

$$\langle f_1, f_2 \rangle := Av(f_1^* \cdot f_2) = \frac{1}{|G|} \sum_{g \in G} f_1(g^{-1}) \cdot f_2(g).$$

Collecting the last two lemmas, we obtain:

**Claim 5.14.** *Let $V$ and $W$ be finite-dimensional $G$-representations. Then we have*

$$\dim \mathrm{Hom}_G(V, W) = \langle \mathrm{ch}_V, \mathrm{ch}_W \rangle.$$

*Proof.* We have $\mathrm{Hom}_G(V, W) = \mathrm{Hom}(V, W)^G$ and therefore

$$\dim \mathrm{Hom}_G(V, W) = \dim \mathrm{Hom}(V, W)^G = Av(\mathrm{ch}_{Hom(V,W)}) = Av(\mathrm{ch}_V^* \cdot \mathrm{ch}_W).$$

□

### 5.1.3

Combining Claim 5.14 and Schur's lemma, we obtain:

**Proposition 5.15** (Orthogonality of characters)**.** *Let $E$ and $F$ be irreducible $G$-representations. Then*

$$\langle \mathrm{ch}_E, \mathrm{ch}_F \rangle = \begin{cases} d_E & \text{if } E \text{ is isomorphic to } F \\ 0 & \text{if } E \text{ is not isomorphic to } F \end{cases},$$

*where $d_E \in \mathbb{Z}_{\geq 1}$ is the dimension of the division algebra* $\mathrm{End}_G(E)$*.*

*Proof.* By Claim 5.14, we have $\langle \mathrm{ch}_E, \mathrm{ch}_F \rangle = \dim \mathrm{Hom}_G(E, F)$. By Schur's lemma, $\mathrm{Hom}_G(E, F)$ is 0 if $E$ is not isomorphic to $F$. If $E$ is isomorphic to $F$ then once we choose an isomorphism $E \cong F$ we obtain an isomorphism $\mathrm{Hom}_G(E, F) \cong \mathrm{Hom}_G(E, E)$, and so the claim is clear. □

Here there is a danger, of $d_E$ being equal to zero in $k$. If $k$ is algebraically closed, the proposition is espeically nice:

**Corollary 5.16.** *Assume that $k$ is algebraically closed. Let $E$ and $F$ be irreducible $G$-representations. Then*

$$\langle \mathrm{ch}_E, \mathrm{ch}_F \rangle = \begin{cases} 1 & \text{if } E \text{ is isomorphic to } F \\ 0 & \text{if } E \text{ is not isomorphic to } F \end{cases}.$$

We also have the following corollary:

**Corollary 5.17.** *Suppose that $k$ is algebraically closed. Let $E_1, \ldots, E_r$ be pairwise non-isomorphic irreducible $G$-representations. Then $\mathrm{ch}_{E_1}, \ldots, \mathrm{ch}_{E_r} \in \mathrm{Fun}_k(G)^{cl}$ are linearly independent.*

*Proof.* If $\sum_i c_i \mathrm{ch}_{E_i} = 0$ for scalars $c_i \in k$ then for a given $1 \leq j \leq r$ we have

$$0 = \langle \mathrm{ch}_{E_j}, \sum_i c_i \mathrm{ch}_{E_i} \rangle = \sum_i c_i \langle \mathrm{ch}_{E_j}, \mathrm{ch}_{E_i} \rangle = c_j.$$

□

### 5.1.4

Now we come to the main theorem about characters:

**Theorem 5.18.** *Assume that $k$ is algebraically closed. Let $E_1, \ldots, E_r$ be an exhaustive family of irreducible $G$-representations (i.e. no two representations in the family are isomorphic and every irreducible representation is isomorphic to one from the family). Then $\mathrm{ch}_{E_1}, \ldots, \mathrm{ch}_{E_r}$ form a basis for $\mathrm{Fun}_k(G)^{cl}$.*

*Proof.* We have already seen that $\mathrm{ch}_{E_1}, \ldots, \mathrm{ch}_{E_r}$ are linearly independent. On the other hand, we have already seen above that $r$, the number of irreducible $G$-representations (up to isomorphism), is equal to the the dimension of $\mathrm{Fun}_k(G)^{cl}$ (which is the number of conjugacy classes in $G$). Therefore our characters must also span $\mathrm{Fun}_k(G)^{cl}$. □

**Remark 5.19.** One can see that if $k$ is not algebraically closed, the characters of an exhaustive list of irreducible representations are still linearly independent. For example this can be seen by base changing to large enough field extension. We omit the details for now (notice that if $k$ is of characteristic zero then this is clear by the same argument as we had above for an algebraically closed $k$).

### 5.1.5

**Exercise 5.2.** *Let $V$ and $W$ be finite-dimensional $G$-representations. Then*

$$\mathrm{ch}_{V \oplus W} = ch_V + ch_W,$$

*where for $f_1, f_2 \in \mathrm{Fun}_k(G)$ we define $f_1 + f_2 \in \mathrm{Fun}_k(G)$ by $(f_1 + f_2)(g) := f_1(g) + f_2(g)$.*

**Claim 5.20.** *Suppose that $k$ has characteristic zero. Let $V$ and $W$ be finite-dimensional $G$-representations. Then $V$ is isomorphic to $W$ (non-canonically) if and only if* $\mathrm{ch}_V = \mathrm{ch}_W$.

*Proof.* If $V$ is isomorphic to $W$ then clearly $\mathrm{ch}_V = \mathrm{ch}_W$. Conversely, suppose that $\mathrm{ch}_V = \mathrm{ch}_W$. To show that $V$ is isomorphic to $W$, it is enough to show that, for every irreducible $G$-representation $E$, we have $[V : E] = [W : E]$ (because then $V$ and $W$ are both isomorphic to direct sums of irreducibles, with each irreducible appearing the same number of times). We have:

$$[V : E] = \dim \mathrm{Hom}_G(V, E) = \langle \mathrm{ch}_V, \mathrm{ch}_E \rangle =$$

$$= \langle \mathrm{ch}_W, \mathrm{ch}_E \rangle = \dim \mathrm{Hom}_G(W, E) = [W : E].$$

This equality is in $k$, but since $k$ has characteristic zero, this is a honest equality of integers. ☐

### 5.1.6

A classical problem, for a given finite group $G$, is to write the "character table" over $\mathbb{C}$. This means to make a table which describes the value of the character of each irreducible representation at each conjugacy class. This implicitly requires to first determine a parametrization of the irreducible representations. For example, for $S_3$ we will have (the horizontal labeling is of conjugacy classes in $S_3$, depicted by the cycle structure, while the vertical labeling is of the irreducible representations, up to isomorphism):

| | $(\bullet)(\bullet)(\bullet)$ | $(\bullet\bullet)(\bullet)$ | $(\bullet\bullet\bullet)$ |
|---|---|---|---|
| $\mathbb{C}_1$ | 1 | 1 | 1 |
| $\mathbb{C}_{sgn}$ | 1 | $-1$ | 1 |
| Standard | 2 | 0 | $-1$ |

### 5.1.7

We assume that $k$ is algebraically closed. A philosophy is that the representation theory of a given finite group $G$ should be "the same" over any algebraically closed field $k$ whose characteristic does not divide $|G|$. From this, one can speculate that given an irreducible $G$-representation $E$ over $k$, the characteristic of $k$ should not divide $\dim_k E$ (because otherwise this would somehow distinguish this characteristic from other ones, contrary to the philosophy). In other words, $\dim_k E$ should not be equal to zero in $k$:

**Claim 5.21.** *Let $E$ be an irreducible $G$-representation over $k$. Then the characteristic of $k$ does not divide* $\dim_k E$.

*Proof.* Let us proceed by way of contradiction. Let $f \in \mathrm{Fun}_k(G)^{cl}$, and let $d := \sum_{g \in G} f(g) \cdot \delta_g \in Z(k[G])$. The action of $d$ on $E$ provides an element $T_d \in \mathrm{End}_G(E)$. By Schur's lemma, $T_d$ is a scalar multiple of the identity. Hence the trace of $T_d$ is a multiple of $\dim_k E$, which we assume to be zero in $k$. On the other hand, we calculate

$$\operatorname{Tr}(T_d) = \sum_{g \in G} f(g) \cdot \operatorname{ch}_E(g) = |G| \cdot \langle f^*, \operatorname{ch}_E \rangle.$$

Taking $f := \operatorname{ch}_E^*$ we obtain that $\langle \operatorname{ch}_E, \operatorname{ch}_E \rangle = 0$, contradicting that we know $\langle \operatorname{ch}_E, \operatorname{ch}_E \rangle = 1$. □

Having established that, if we believe the philosophy, we should then speculate that $E$ has "existence" over all fields $k$ as mentioned above, and therefore that the primes that divide the dimension of $E$ should divide $|G|$, in other words we speculate that the dimension of $E$ should divide a power of $|G|$. In fact, we will see later that even more is true, namely that the dimension of $E$ divides $|G|$ itself.

### 5.1.8

We again assume that $k$ is algebraically closed. Recall that we defined, for an irreducible $G$-representation $E$, an idempotent $e_E \in Z(k[G])$, which acts by identity on $E$ and by 0 on any irreducible $G$-representation which is not isomorphic to $E$. An alternative characterization is that $e_E$ is the unique element in $k[G]$ whose action on every finite-dimensional $G$-representation $V$ is the $G$-morphic projection onto $V_E$. When $G$ is commutative, we gave a formula for $e_E$ before. We now consider the general case.

**Claim 5.22.**

$$e_E = \frac{\dim_k E}{|G|} \sum_{g \in G} \operatorname{ch}_E(g^{-1}) \cdot \delta_g.$$

*Proof.* Let $F$ be an irreducible $G$-representation, and write $\rho : k[G] \to \operatorname{End}_k(F)$ for the corresponding homomorphism. By Schur's lemma, $\rho(e_E)$ is a scalar multiple of the identity. Since we showed that $\dim_k F$ is not equal to 0 in $k$, it is enough to show therefore that $\operatorname{Tr}(\rho(e_E))$ is equal to 0 if $F$ is not isomorphic to $E$ and to $\dim_k(E)$ if $F$ is isomorphic to $E$. We have:

$$\operatorname{Tr}(\rho(e_E)) = \frac{\dim_k E}{|G|} \sum_{g \in G} \operatorname{ch}_E(g^{-1}) \cdot \operatorname{Tr}(\rho(g)) = \frac{\dim_k E}{|G|} \sum_{g \in G} \operatorname{ch}_E(g^{-1}) \cdot \operatorname{ch}_F(g) =$$

$$= \dim_k E \cdot \langle \operatorname{ch}_E, \operatorname{ch}_F \rangle,$$

and so the desired follows from orthogonality relations for characters. □

### 5.1.9

Let us here also make a comment regarding the case $k = \mathbb{C}$. Then one can also consider a Hermitian form $\langle -, - \rangle_{hr}$ on $\mathrm{Fun}_{\mathbb{C}}(G)$:

$$\langle f_1, f_2 \rangle_{hr} := \frac{1}{|G|} \sum_{g \in G} f_1(g) \cdot \overline{f_2(g)}.$$

This form is non-degenerate (i.e. a Hermitian inner product). We claim that for finite-dimensional irreducible $G$-representations $V, W$ we have

$$\langle \mathrm{ch}_V, \mathrm{ch}_W \rangle = \langle \mathrm{ch}_V, \mathrm{ch}_W \rangle_{hr}.$$

In fact, this follows from the relation $\mathrm{ch}_W(g^{-1}) = \overline{\mathrm{ch}_W(g)}$. To see this relation, let us consider the eigenvalues (taken with algebraic multiplicity) $(\lambda_i)$ of the action of $g$ on $W$. Those are all roots of unity (since $g$ lies in a finite group, and hence becomes 1 when raised to some power). The eigenvalues of the action of $g^{-1}$ on $W$ are $(\lambda_i^{-1})$. We therefore get

$$\mathrm{ch}_W(g^{-1}) = \sum_i \lambda_i^{-1} = \sum_i \overline{\lambda_i} = \overline{\sum_i \lambda_i} = \overline{\mathrm{ch}_W(g)}.$$

The bilinear product is better in that it is the one that generalizes to other fields; The Hermitian product is better in that it is the one that generalizes to representations of topological groups on Hilbert spaces.

### 5.1.10

A simple useful lemma which I didn't find place/use for:

**Lemma 5.23.** *The symmetric bilinear form $\langle -, - \rangle$ defined on $\mathrm{Fun}_k(G)$ is non-degenerate, and its restriction to $\mathrm{Fun}_k(G)^{cl}$ is also non-degenerate.*

*Proof.* Let $f \in \mathrm{Fun}_k(G)$. If $\langle f, f' \rangle = 0$ for all $f' \in \mathrm{Fun}_k(G)$, then in particular $\langle f, \delta_g^* \rangle = 0$ for every $g \in G$, where $\delta_g$ being the function that equals to 1 at $g$ and to 0 at the rest of points. But we have $\langle f, \delta_g^* \rangle = \frac{1}{|G|} f(g)$, so $f(g) = 0$ for all $g \in G$, i.e. $f = 0$. This showed that $\langle, -, - \rangle$ is non-degenerate on $\mathrm{Fun}_k(G)$.

We want to show now that the restriction of $\langle -, - \rangle$ to $\mathrm{Fun}_k(G)^{cl}$ is also non-degenerate. Let us consider the linear operator

$$Av : \mathrm{Fun}_k(G) \to \mathrm{Fun}_k(G)$$

given by

$$Av(f)(g) := \frac{1}{|G|} \sum_{h \in G} f(hgh^{-1}).$$

One checks immediately that $Av$ is a projection operator on $\mathrm{Fun}_k(G)^{cl}$ (in fact, it is just oue usual projection operator, with respect to the action of $G$ on $\mathrm{Fun}_k(G)$ by conjugation). We also have the property

$$\langle Av(f), f' \rangle = \langle f, Av(f') \rangle$$

for $f, f' \in \mathrm{Fun}_k(G)$. Let now $f \in \mathrm{Fun}_k(G)^{cl}$ be such that $\langle f, f' \rangle = 0$ for all $f' \in \mathrm{Fun}_k(G)^{cl}$. Then for every $f' \in \mathrm{Fun}_k(G)$ we have

$$\langle f, f' \rangle = \langle Av(f), f' \rangle = \langle f, Av(f') \rangle = 0$$

and so $f = 0$ by the non-degeneracy of $\langle -, - \rangle$ on $\mathrm{Fun}_k(G)$. □

## 5.2 Integrality

Throughout this subsection, we assume that $k$ is algebraically closed.

### 5.2.1

Let us recall the basics of integrality of elements in a ring.

**Definition 5.24.** Let $R$ be a ring and $a \in R$. We say that $a$ is **integral** (more precisely, $\mathbb{Z}$-integral) if there exists a monic polynomial $f \in \mathbb{Z}[X]$ such that $f(a) = 0$.

**Remark 5.25.** Let $R$ be a ring and $S \subset R$ a subring. Let $a \in S$. Clearly $a$ is integral as an element of $S$ if and only if $a$ is integral as an element of $R$. Therefore we will sometimes be loose and simply speak of $a$ being integral, without recourse to the containing ring.

**Lemma 5.26.** *Let $R$ be a ring and $a \in R$. Then $a$ is integral if and only if the $\mathbb{Z}[a] \subset R$, the $\mathbb{Z}$-span of $\{1, a, a^2, \ldots\}$, is a finitely generated $\mathbb{Z}$-module.*

*Proof.* Suppose that $a$ is integral and $f \in \mathbb{Z}[X]$ a monic polynomial, say of degree $d$, such that $f(a) = 0$. Then clearly $a^d$ lies in the $\mathbb{Z}$-span of $\{1, a, \ldots, a^{d-1}\}$. Then, by multiplying by $a$, we see that $a^{d+1}$ lies in the $\mathbb{Z}$-span of $\{a, \ldots, a^d\}$ and so in the $\mathbb{Z}$-span of $\{1, \ldots, a^{d-1}\}$, and continuing like this by induction we see that $\mathbb{Z}[a]$ lies in the $\mathbb{Z}$-span of $\{1, \ldots, a^{d-1}\}$, so equal to it. Therefore it is a finitely generated $\mathbb{Z}$-module.

Now suppose conversely that $\mathbb{Z}[a]$ is a finitely generated $\mathbb{Z}$-module. Then clearly there exists $d \in \mathbb{Z}_{\geq 1}$ such that $\{1, \ldots, a^{d-1}\}$ $\mathbb{Z}$-spans $\mathbb{Z}[a]$, and therefore in particular $a^d$ can be expressed as a $\mathbb{Z}$-linear combination of the elements $\{1, \ldots, a^{d-1}\}$, say $a^d = c_0 \cdot 1 + \ldots + c_{d-1} \cdot a^{d-1}$, so $f(X) := (-c_0) \cdot 1 + \ldots + (-c_{d-1}) \cdot X^{d-1} + X^d$ is a monic polynomial satisfying $f(a) = 0$, as desired. □

**Corollary 5.27.** *Let $R$ be a ring, and suppose that $R$ is finitely generated as a $\mathbb{Z}$-module. Then all elements in $R$ are integral.*

*Proof.* Use the previous lemma, recalling in addition that $\mathbb{Z}$ is Noethreian- a submodule of a finitely generated $\mathbb{Z}$-module is again finitely generated. ☐

**Claim 5.28.** *Let $R$ be a commutative ring. Then the subset of integral elements in $R$ is a subring.*

*Proof.* Clearly $0, 1 \in R$ are integral. Let $a, b \in R$ be two integral elements, say $a^d$ lies in the $\mathbb{Z}$-span of $\{1, a, \ldots, a^{d-1}\}$ and $b^e$ lies in the $\mathbb{Z}$-span of $\{1, b, \ldots, b^{e-1}\}$. Then it is immediate to see that $\mathbb{Z}[a, b]$ (the $\mathbb{Z}$-span of $\{a^n b^m\}_{n,m\in\mathbb{Z}_{\geq 0}}$) is generated as a $\mathbb{Z}$-module by $\{a^n b^m\}_{0\leq n\leq d-1, 0\leq m\leq e-1}$ (notice how commutativity is used here), and so is a finitely generated $\mathbb{Z}$-module. Therefore by the previous corollary $a + b$, $a - b$ and $ab$, lying in $\mathbb{Z}[a, b]$, are integral. ☐

**Exercise 5.3.** *The subring of integral elements in $\mathbb{Q}$ is $\mathbb{Z}$.*

**Remark 5.29.** Let us also notice the obvious property, that if $\phi : R \to S$ is a morphism of rings and $a \in R$ is an integral element, then $\phi(a)$ is an integral element.

### 5.2.2

**Lemma 5.30.**

1. *Let $d = \sum_{g\in G} c_g \cdot \delta_g \in k[G]$ and assume that all $c_g$ are integers. Then $d$ is integral.*

2. *Let $d = \sum_{g\in G} c_g \cdot \delta_g \in Z(k[G])$. If $c_g$ are integral elements in $k$ then $d$ is an integral element in $Z(k[G])$.*

*Proof.*

1. Our element sits in the image of the obvious morphism of rings $\mathbb{Z}[G] \to k[G]$, so the claim follows from all elements of $\mathbb{Z}[G]$ being integral, which in turn follows from $\mathbb{Z}[G]$ being a finitely generated $\mathbb{Z}$-module.

2. Since $Z(k[G])$ is commutative, the integral elements in it form a subring. Since our $d$ is a sum of products of integral elements $c_g$ by integral elements $z_C$ (the latter are integral by the previous item), the claim follows.

☐

**Lemma 5.31.** *Let $V$ be a finite-dimensional $G$-representation. Then for every $g \in G$, the element $\mathrm{ch}_V(g) \in k$ is integral.*

*Proof.* Notice that since $g$ to some power is equal to 1, so are the eigenvalues of $g \curvearrowright V$. Therefore all these eigenvalues are roots of unity, and thus clearly integral. Therefore their sum, which is $\mathrm{ch}_V(g) = \mathrm{Tr}(g \curvearrowright V)$, is integral. ☐

### 5.2.3

Here is the strengthening of Claim 5.21 that we promised:

**Proposition 5.32.** *Let $E$ be an irreducible $G$-representation. Then* $\dim E$ *divides* $|G|$.

*Proof.* We will only prove the proposition assuming that the characteristic of $k$ is 0 (check if one can eliminate this simply).

Let us denote by $\rho : k[G] \to \mathrm{End}(E)$ the $k$-algebra morphism corresponding to the $G$-action on $E$. Recall the central idempotent

$$e_E = \frac{\dim E}{|G|} \sum_{g \in G} \mathrm{ch}_E(g^{-1}) \cdot \delta_g \in Z(k[G]).$$

We saw that it acts as identity on $E$ (in particular, recall that $\dim E$ is non-zero in $k$). Therefore

$$d := \sum_{g \in G} \mathrm{ch}_E(g^{-1}) \cdot \delta_g \in Z(k[G])$$

acts as multiplication by the scalar $\lambda := \frac{|G|}{\dim E}$. Notice that $d$ is integral by the above lemmas, and therefore $\rho(d) = \lambda \cdot \mathrm{Id}_E$ is integral in $k \cdot \mathrm{Id}_E \subset \mathrm{End}(E)$. Therefore $\lambda \in k$, which is identified with $\rho(d)$ via $k \cong k \cdot \mathrm{Id}_E$, is an integral element. So $\frac{|G|}{\dim E}$ is an integral element in $\mathbb{Q} \subset k$, and so lies in $\mathbb{Z}$, meaning that $\dim E$ divides $|G|$.

□

### 5.2.4

In fact, we can now, based on the previous result, formulate a stronger one:

**Proposition 5.33.** *Let $E$ be an irreducible $G$-representation. Then* $\dim E$ *divides* $[G : Z(G)]$.

**Remark 5.34.** In fact, later we will see that given a normal abelian subgroup $A \subset G$, the dimension of any irreducible representation divides $[G : A]$.

To proof this proposition, we will need the following remark and exercise:

**Remark 5.35.** Let $H_1$ and $H_2$ be finite groups and let $V_1$ and $V_2$ be representations of $H_1$ and $H_2$. We can make $V_1 \otimes_k V_2$ a representation of $H_1 \times H_2$ by letting $(h_1, h_2)$ acts by the endomorphism $V_1 \otimes_k V_2 \to V_1 \otimes_k V_2$ characterized by $v_1 \otimes v_2 \mapsto h_1 v_1 \otimes h_2 v_2$ (as always, this endomorphism exists by the universal property of the tensor product as we have a bilinear form $(v_1, v_2) \mapsto h_1 v_1 \otimes_k h_2 v_2$). Before, if we had representations $V_1$ and $V_2$ of a single group $H$, we made $V_1 \otimes_k V_2$ an $H$-representation. What is the relation? Here we made $V_1 \otimes_k V_2$ an $(H \times H)$-representation. Restricting along the diagonal homomorphism $H \to H \times H$ (given by $h \mapsto (h, h)$), we obtain our $H$-representation.

**Exercise 5.4.** *Let $H_1$ and $H_2$ be finite groups and let $E_1$ and $E_2$ be irreducible representations of $H_1$ and $H_2$. Then $E_1 \underset{k}{\otimes} E_2$ is an irreducible representation of $H_1 \times H_2$.*

*Proof (Attributed by Serre to Tate).* Let us write $A := Z(G)$ for brevity. Let $m \geq 1$ and consider $E^{\otimes m} = E \otimes_k E \otimes_k \ldots \otimes_k E$ as a representation of $G^m = G \times G \times \ldots \times G$. By the above exercise, $E^{\otimes m}$ is an irreducible representation of $G^m$. Let us consider the subgroup $A_m \subset A^m$ consisting of $(a_1, \ldots, a_m)$ satisfying $a_1 \cdot a_2 \cdot \ldots \cdot a_m = 1$. Since, by Schur's lemma, $A$ acts on $E$ by scalars, so via a character $\chi : A \to k^\times$ (i.e. $av = \chi(a)v$ for $a \in A, v \in E$), we see that $A_m$ acts trivially on $E^{\otimes m}$ ($(a_1, \ldots, a_m)$ acts by $\chi(a_1) \cdot \ldots \cdot \chi(a_m) = \chi(a_1 \cdot \ldots \cdot a_m) = \chi(1) = 1$). Therefore, we can consider $E^{\otimes m}$ as a representation of $G^m/A_m$, still irreducible. Hence $(\dim E)^m$ divides $|G|^m/|A|^{m-1}$. Hence, given a prime $p$ and denoting, for an integer $n$, by $v_p(n)$ the amount of times $p$ enters $n$, we have $mv_p(\dim E) \leq mv_p(|G|) - (m-1)v_p(|A|)$. Therefore $v_p(\dim E) \leq v_p(|G|/|A|) + \frac{1}{m} v_p(|A|)$. Taking $m \to \infty$, we obtain $v_p(\dim E) \leq v_p(|G|/|A|)$. As this holds for every prime $p$, we get that $\dim E$ divides $|G|/|A|$. □

## 5.3 Burnside's theorem

In this subsection, we prove Burnside's theorem:

**Theorem 5.36** (Burnside, 1904)**.** *If $|G|$ is divisible by at most two prime numbers, then $G$ is solvable.*

This is a very nice illustration of representation theory of finite groups, since the theorem statement does not mention representations at all, but the proof will use representation theory.

### 5.3.1

For the next lemma, let us notice that an **algebraic integer** in $\mathbb{C}$ is simply a different terminology for an integral element in $\mathbb{C}$ in our above sense.

**Lemma 5.37.** *Let $\zeta_1, \ldots, \zeta_n \in \mathbb{C}^\times$ be roots of unity. Then:*

1. *The average $\frac{\zeta_1 + \ldots + \zeta_n}{n}$ has absolute value in $[0, 1]$, and $1$ is attained if and only if $\zeta_1 = \zeta_2 = \ldots = \zeta_n$.*
2. *The average $\frac{\zeta_1 + \ldots + \zeta_n}{n}$ is an algebraic integer if and only if it is equal to $0$ or $\zeta_1 = \zeta_2 = \ldots = \zeta_n$.*

*Proof.* Item (1) is easy to imagine visually, we leave it as an exercise. Let us show (2). By (1), it is enough to see that if our average $a$ is an algebraic integer if and only if its absolute value is either 0 or 1. Here we use very mild field theory. Recall that we have conjugates of $a$ (the elements in $\mathbb{C}$ which are obtained as images of $a$ under homomorphisms $\mathbb{Q}[a] \to \mathbb{C}$ of field extensions over $\mathbb{Q}$), and that the product $N(a)$ of these conjugates lies in $\mathbb{Q}$. Then clearly all such conjugates are also algebraic integers, and hence their product $N(a)$ is an algebraic integer, and therefore simply an integer, as it lies in $\mathbb{Q}$. Also, notice that all the conjugates are also averages of roots of unity, and therefore by (1) their absolute values lie in $[0,1]$, and so the absolute values of their product $N(a)$ lies in $[0,1]$. Since $N(a)$ is an integer, we obtain that its absolute value is either 0 or 1. In the first case we get $a = 0$. In the second case, we get $|a| = 1$. □

### 5.3.2

From this previous lemma we will deduce the following one:

**Lemma 5.38.** *Let $E$ be an irreducible $G$-representation over $\mathbb{C}$. Let $g \in G$ and suppose that $|C_g|$ and $\dim E$ are relatively prime. Then either $\mathrm{ch}_E(g) = 0$ or $g$ acts on $E$ by a scalar.*

*Proof.* Denote by $\rho : k[G] \to \mathrm{End}(E)$ the morphism corresponding to $E$ being a $G$-representation. Denote $d := \sum_{h \in C_g} \delta_h \in Z(k[G])$. Then $\rho(d)$ is a scalar multiply of the identity by Schur's lemma. Since $d$ is an integral element, its image under $\rho$ in $\mathbb{C} \cong \mathbb{C} \cdot \mathrm{Id}_E \subset \mathrm{End}(E)$ is an integral element, and this image is

$$\frac{\mathrm{Tr}(\rho(d))}{\dim E} = \frac{|C_g| \cdot \mathrm{ch}_E(g)}{\dim E}.$$

Since $|C_g|$ is relatively prime to $\dim E$, we deduce that $\frac{\mathrm{ch}_E(g)}{\dim E}$ is integral (a small exercise - use that 1 can be expressed as a $\mathbb{Z}$-linear combination of $|C_g|$ and $\dim E$). Now notice that $\frac{\mathrm{ch}_E(g)}{\dim E}$ is the average of the eignevalues of $\rho(g)$, and therefore by the previous lemma, we must have that either $\mathrm{ch}_E(g) = 0$ or all the eigenvalues of $\rho(g)$ are equal, in which case $\rho(g)$ is a scalar multiple of the identity, as $\rho(g)$ is diagnolizable. □

### 5.3.3

From this last lemma, we deduce the following claim, in the statement of which there is no mention of representation theory (!)[5]:

**Claim 5.39.** *If $G$ contains a conjugacy class in which the number of elements is a positive power of a prime number, then $G$ is not simple.*

*Proof.* Let $C_g \subset G$ be a conjugacy class whose order is a positive power of the prime $p$. We will work with representations over $\mathbb{C}$. It suffices to show that there exists a non-trivial irreducible $G$-representation $E$ on which elements in $C_g$ act by scalar (then taking two different elements $g, h \in C_g$ we will have that $gh^{-1}$ acts by identity, and so the kernel of our $\rho : G \to \mathrm{End}(E)$ will be a normal subgroup in $G$ which is nor $G$ neither $\{1\}$). Using the previous lemma, it is enough to to show that there exists a non-trivial irreducible $G$-representation

[5] But one can say that representation theory always waited to appear, as the spectral dual for our non-commutative group - it is not an "unnatural over-construct".

$E$ such that $p$ does not divide $\dim E$ and $\mathrm{ch}_E(g) \neq 0$. We have the following orthogonality relation:

$$\sum_{\text{irreducible } E} (\dim E) \cdot \mathrm{ch}_E(g) = 0$$

(where we run over all irreducible representations, up to isomorphism). We break it as follows:

$$1 + \sum_{\substack{E \text{ s.t. } p \\ \text{divides } \dim E}} (\dim E) \cdot \mathrm{ch}_E(g) + \sum_{\substack{\text{non-trivial } E \text{ s.t. } p \\ \text{does not divide } \dim E}} (\dim E) \cdot \mathrm{ch}_E(g) = 0.$$

We can consider this as an equation in the ring of algebraic integers, and talk about divisibility in this ring. Since $p$ is not a unit (does not divide 1), we see that there exists a non-trivial $E$ such that $p$ does not divide $(\dim E) \cdot \mathrm{ch}_E(g)$, in particular $p$ does not divide $\dim E$ and $(\dim E) \cdot \mathrm{ch}_E(g) \neq 0$, so $\mathrm{ch}_E(g) \neq 0$, as desired. □

#### 5.3.4

We now can prove Burnside's theorem, Theorem 5.36.

*Proof (of Theorem 5.36).* From a basic course in group theory, it is known that groups of prime power order are solvable. The theorem is therefore equivalent to showing that there are no simple groups whose order is divisible by exactly two primes. Let therefore $H$ be a finite group whose order is divisible by exactly two primes, $p$ and $q$. If $Z(H)$ is not trivial, then $H$ is not simple and we are done. Assume therefore that $|Z(H)| = 1$. Then The $pq$ does not divide the sum of numbers of elements in all the non-trivial conjugacy classes. Therefore there must be a conjugacy class in which the number of elements is a power of $p$ or a power of $q$, and then by the previous claim $H$ is not simple. □

# 6 Induction

## 6.1 Categories and functors

We will familiarize ourselves with the basic language of category theory, because it seems a bit lacking to talk about induction without mentioning adjoint functors.

#### 6.1.1

A **($k$-linear) category** is a collection $\mathcal{C}$ (elements of the collection are called **objects**) together with, for any two objects $M, N \in \mathcal{C}$, a $k$-vector space $\mathrm{Hom}(M,N) = \mathrm{Hom}_{\mathcal{C}}(M,N)$ (called the ***Hom*-space**, or the **space of morphisms**), and for any three objects $M, N, L \in \mathcal{C}$ a $k$-bilinear map

$$\mathrm{Hom}(N,L) \times \mathrm{Hom}(M,N) \to \mathrm{Hom}(M,L)$$

(called the **composition**), such that composition is associative and has units $\mathrm{id}_M \in \mathrm{Hom}(M,M)$ (we let the reader write down the meaning of this accurately).

The datum $\alpha \in \mathrm{Hom}(M,N)$ we also write $\alpha : M \to N$. The composition of $(\beta, \alpha) \in \mathrm{Hom}(N,L) \times \mathrm{Hom}(M,N)$ we denote $\beta \circ \alpha \in \mathrm{Hom}(M,L)$.

### 6.1.2

Given a group $G$ and a field $k$, we can form the $k$-linear category $\mathrm{Rep}(G)$ of $G$-representations over $k$ (if we wamt to be more precise, we can denote it $\mathrm{Rep}_k(G)$). Here objects are $G$-representations over $k$, and for two $G$-representations over $k$ the space $\mathrm{Hom}_{\mathrm{Rep}(G)}(M,N)$ is our $\mathrm{Hom}_G(M,N)$. Composition is defined in the obvious way, as composition of transformations.

Similarly, given a $k$-algebra $A$, we can consider the $k$-linear category $\mathrm{Mod}(A)$ of (left) $A$-modules.

### 6.1.3

Let $\mathcal{C}$ and $\mathcal{D}$ be $k$-linear categories. A **($k$-linear) functor** $F : \mathcal{C} \to \mathcal{D}$ consists of associating to any object $M \in \mathcal{C}$ an object $F(M) \in \mathcal{D}$ and also, to objects $M, N \in \mathcal{C}$ associating a $k$-linear map$\mathrm{Hom}_{\mathcal{C}}(M,N) \to \mathrm{Hom}_{\mathcal{D}}(F(M), F(N))$ (given $\alpha : M \to N$ we denote by $F(\alpha) : F(M) \to F(N)$ the image under this map). This should satisfy the rules $F(\mathrm{id}_M) = \mathrm{id}_{F(M)}$ and $F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$.

We will only deal with $k$-linear functors between $k$-linear categories, so we might omit the "$k$-linear" adjective, meaning it implicitly.

The datum of a functor $F$ from $\mathcal{C}$ to $\mathcal{D}$ we also write $F : \mathcal{C} \to \mathcal{D}$.

### 6.1.4

Let $F, G : \mathcal{C} \to \mathcal{D}$. A **morphism** $\sigma : F \to G$ is the data of, for any $M \in \mathcal{C}$, a morphism $\sigma_M : F(M) \to G(M)$, such that, for any morphism $\alpha : M \to N$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} F(M) & \xrightarrow{\sigma_M} & G(M) \\ \downarrow{\scriptstyle F(\alpha)} & & \downarrow{\scriptstyle G(\alpha)} \\ F(N) & \xrightarrow{\sigma_N} & G(N) \end{array}$$

(the commutation of the diagram means $G(\alpha) \circ \sigma_M = \sigma_N \circ F(\alpha)$).

### 6.1.5

Let $\mathcal{C}$ and $\mathcal{D}$ be $k$-linear categories and $F : \mathcal{C} \to \mathcal{D}$ a $k$-linear functor. Fix $U \in \mathcal{D}$. We can talk about morphisms from objects $F(M)$ (where $M \in \mathcal{C}$) to $U$. Is there a "universal" one? That would be an object $\widetilde{U} \in \mathcal{C}$ equipped with a morphism $\beta^{univ} : F(\widetilde{U}) \to U$, such that, for any object $M \in \mathcal{C}$ equipped with a morphism $\beta : F(M) \to U$, there exists a unique morphism $\alpha : M \to \widetilde{U}$ such that $\beta = \beta^{univ} \circ F(\alpha)$. Such a pair $(\widetilde{U}, \beta^{univ})$ would be unique in the following sense. Given two such pairs $(\widetilde{U}_1, \beta_1^{univ})$ and $(\widetilde{U}_2, \beta_2^{univ})$, by the universal property of $\widetilde{U}_1$ we obtain a morphism $\iota_{21} : \widetilde{U}_2 \to \widetilde{U}_1$ (the unique one such that $\beta_1^{univ} \circ F(\iota_{21}) = \beta_2^{univ}$). Similarly, using the universal property of $\widetilde{U}_2$, we obtain a morphism $\iota_{12} : \widetilde{U}_1 \to \widetilde{U}_2$ (the unique one such that $\beta_2^{univ} \circ F(\iota_{12}) = \beta_1^{univ}$). Then, since $\beta_2^{univ} \circ F(\iota_{12} \circ \iota_{21}) = \beta_2^{univ}$ and also $\beta_2^{univ} \circ F(\mathrm{id}_{\widetilde{U}_2}) = \beta_2^{univ}$, we see that we must have, by the uniqueness part of the universal property of $\widetilde{U}_2$, that $\iota_{12} \circ \iota_{21} = \mathrm{id}_{\widetilde{U}_2}$. Similarly, we obtain that $\iota_{21} \circ \iota_{12} = \mathrm{id}_{\widetilde{U}_1}$. Hence, we constructed a specific isomorphism between $\widetilde{U}_1$ and $\widetilde{U}_2$ (characterized as the unique isomorphism $\iota_{12} : \widetilde{U}_1 \to \widetilde{U}_2$ satisfying $\beta_2^{univ} \circ F(\iota_{12}) = \beta_1^{univ}$).

Notice that we can also formulate the universal property by saying that, given $M \in \mathcal{C}$, the map

$$\mathrm{Hom}_{\mathcal{D}}(M, \widetilde{U}) \xrightarrow{\beta^{univ} \circ F(-)} \mathrm{Hom}_{\mathcal{C}}(F(M), U)$$

is a bijection.

Suppose that for any $U \in \mathcal{D}$ there exists such a universal $(\widetilde{U}, \beta_U^{univ})$. Given a morphism $\beta : U \to V$, we can consider the composition $F(\widetilde{U}) \xrightarrow{\beta_U^{univ}} U \xrightarrow{\beta} V$. By the universal property of $\widetilde{V}$, there exists a unique morphism $\alpha : \widetilde{U} \to \widetilde{V}$ such that

$$\begin{array}{ccc} F(\widetilde{U}) & \xrightarrow{\beta_U^{univ}} & U \\ {\scriptstyle F(\alpha)}\downarrow & & \downarrow{\scriptstyle \beta} \\ F(\widetilde{V}) & \xrightarrow{\beta_V^{univ}} & V \end{array}$$

commutes.

We now define a functor $\mathcal{C} \leftarrow \mathcal{D} : F^r$ as follows. We set $F^r(U) := \widetilde{U}$ and for a morphism $\beta : U \to V$ we set $F^r(\beta) : \widetilde{U} \to \widetilde{V}$ to be $\alpha$ as just described.

The functor $F^r$ is called **the right adjoint of** $F$.

We can characterize $F^r$ "in its totality" as follows. We are also given a morphism of functors $\beta^{univ} : F \circ F^r \to \mathrm{Id}_{\mathcal{D}}$ and demand that for $M \in \mathcal{C}$ and $U \in \mathcal{D}$ we have that

$$\mathrm{Hom}_{\mathcal{C}}(M, F^r(U)) \xrightarrow{F(-) \circ \beta_U^{univ}} \mathrm{Hom}_{\mathcal{D}}(F(M), U)$$

is a bijection.

### 6.1.6

To summarize, one should internalize the following. Given $F : \mathcal{C} \to \mathcal{D}$, a **right adjoint** of $F$ is a functor $\mathcal{C} \leftarrow \mathcal{D} : F^r$ together with a morphism of functors $\beta^{univ} : F \circ F^r \to \mathrm{Id}_{\mathcal{D}}$, such that for $M \in \mathcal{C}$ and $U \in \mathcal{D}$ we have that

$$\mathrm{Hom}_{\mathcal{C}}(M, F^r(U)) \xrightarrow{F(-)\circ\beta_U^{univ}} \mathrm{Hom}_{\mathcal{D}}(F(M), U)$$

is a bijection. Given two such right adjoints, one constructs a canonical isomorphims between them. In this sense the right adjoint is well defined, if it exists. Or, one can say that it is well defined, whether it exists or not (and existence would be an extra property it might have).

### 6.1.7

Similarly, one defines a **left adjoint**. Namely, given $\mathcal{C} \leftarrow \mathcal{D} : G$, a **left adjoint** of $G$ is a functor $G^l : \mathcal{C} \to \mathcal{D}$ together with a morphism of functors $\beta^{univ} : \mathrm{Id}_{\mathcal{C}} \to G \circ G^l$, such that for $M \in \mathcal{C}$ and $U \in \mathcal{D}$ we have that

$$\mathrm{Hom}_{\mathcal{D}}(G^l(M), U) \xrightarrow{\beta_M^{univ}\circ G(-)} \mathrm{Hom}_{\mathcal{C}}(M, G(U))$$

is a bijection.

### 6.1.8

One can give an equivalent symmetric definition (showing that if $F$ is left adjoint to $G$ then also $G$ is right adjoint to $F$) as follows. Given $F : \mathcal{C} \to \mathcal{D}$ and $\mathcal{C} \leftarrow \mathcal{D} : G$, an **adjunction between $F$ and $G$** (**realizing $F$ as the left adjoint of $G$ and $G$ as the right adjoint of $F$**) is the data, for $M \in \mathcal{C}$ and $U \in \mathcal{D}$, of a $k$-linear isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(F(M), U) \xrightarrow{\gamma_{M,U}} \mathrm{Hom}_{\mathcal{C}}(M, G(U))$$

which satisfies the following two "functoriality" properties. Given a morphism $\alpha : M_1 \to M_2$, the following diagram should commute:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}(F(M_2), U) & \xrightarrow{\gamma_{M_2,U}} & \mathrm{Hom}_{\mathcal{C}}(M_2, G(U)) \\
{\scriptstyle -\circ F(\alpha)}\downarrow & & \downarrow{\scriptstyle -\circ\alpha} \\
\mathrm{Hom}_{\mathcal{D}}(F(M_1), U) & \xrightarrow{\gamma_{M_1,U}} & \mathrm{Hom}_{\mathcal{C}}(M_1, G(U))
\end{array} .$$

Similarly, given a morphism $\beta : U_1 \to U_2$, an analogous diagram, which we leave to the reader, should commute.

It is an exercise then to relate this to the above, i.e. to extract the morphisms $\mathrm{Id}_{\mathcal{C}} \to G \circ F$ and $F \circ G \to \mathrm{Id}_{\mathcal{D}}$, and so on.

In fact, I plan perhaps to skip in class the previous definition of adjoint functors, and simply give the last one.

### 6.1.9

Let us give an example of an adjunction. Let $\iota : B \to A$ be a morphism of $k$-algebras. We have the **restriction functor**, or **forgetful functor** $\mathrm{res}^A_B$ : $\mathrm{Mod}(A) \to \mathrm{Mod}(B)$ which is the following. Given an $A$-module $M$, we can consider it as a $B$-module by precomposing $A \to \mathrm{End}_k(M)$ with $\iota : B \to A$. If $M \to N$ is a morphism of $A$-modules, it will also be a morphism of $B$-modules (when those are viewed as $B$-modules as just described). In this way we obtain our functor. So, this is a “stupid”, or “easy”, or “straight-forward” functor.

Now the philosophy dictates that we can ask whether $\mathrm{res}^A_B$ has a left adjoint.

Consider the functor $A \underset{B}{\otimes} - : \mathrm{Mod}(B) \to \mathrm{Mod}(A)$. It sends a $B$-module $M$ to the $A$-modules $A \underset{B}{\otimes} M$. If we have a morphism $M \to N$ of $B$-modules, we construct a morphism of $A$-modules $\alpha : A \underset{B}{\otimes} M \to A \underset{B}{\otimes} N$ by sending $a \otimes m$ to $a \otimes \alpha(m)$ for $a \in A, m \in M$ (formalize for yourself the existence and uniqueness of a morphism like that, using the universal property of the tensor product).

We claim that we can set $A \underset{B}{\otimes} -$ and $\mathrm{res}^A_B$ in an adjunction. Namely, given $M \in \mathrm{Mod}(B)$ and $N \in \mathrm{Mod}(A)$, we define an isomorphism

$$\mathrm{Hom}_{\mathrm{Mod}(A)}(A \underset{B}{\otimes} M, N) \to \mathrm{Hom}_{\mathrm{Mod}(B)}(M, \mathrm{res}^A_B(N))$$

as follows. Given a morphism of $A$-modules $\alpha : A \underset{B}{\otimes} M \to N$, we define a morphism of $B$-modules $\beta : M \to N$ by $\beta(m) := \alpha(1 \otimes m)$. We can also define a map in the other direction: Given a morphism of $B$-modules $\beta : M \to N$ we define a morphism of $A$-modules $\alpha : A \underset{B}{\otimes} M \to N$ by $\alpha(a \otimes m) := a\beta(m)$ for $a \in A, m \in M$ (check for yourself that $\alpha$ is uniquely defined by this using the universal property of the tensor product). One checks that these two are mutually inverse, yielding the desired bijection. One then checks that it is functorial as desired.

To summarize, base change is the left adjoint of the forgetful functor, and it might seem more sophisticated than the forgetful functor, but the philosophy is that they contain exactly the same information - each one is determined by the other by adjunction.

### 6.1.10

We also have a right adjoint of the forgetful functor. Namely, we consider the functor $\mathrm{Mod}(B) \to \mathrm{Mod}(A)$ given by $M \mapsto \mathrm{Hom}_B(A, M)$. Here $A$ is considered as a left $B$-module in the usual way via $\iota$. $\mathrm{Hom}_B(A, M)$ is a left $A$-module by setting $(a\phi)(a') := \phi(a'a)$. Here we say that it is a functor without explaining what it does to morphisms - it is assumed that the student checks how to fill in all this (if not internalized this already).

Now we should provide a functorial isomorphism, given $M \in \mathrm{Mod}(B)$ and $N \in \mathrm{Mod}(A)$:

$$\mathrm{Hom}_{\mathrm{Mod}(B)}(\mathrm{res}^A_B(N), M) \to \mathrm{Hom}_{\mathrm{Mod}(A)}(N, \mathrm{Hom}_B(A, M)).$$

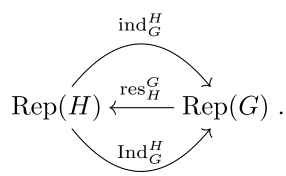
To a morphism of $B$-modules $\beta : N \to M$ we associate a morphism of $A$-modules $\alpha : N \to \mathrm{Hom}_B(A, M)$ given by $\alpha(n)(a) = \beta(an)$. To a morphism of $A$-modules $\alpha : N \to \mathrm{Hom}_B(A, M)$ we associate a morphism of $B$-modules $\beta : N \to M$ given by $\beta(n) := \alpha(n)(1)$. One checks that these two are mutually inverse, yielding the desired bijection. One then checks that it is functorial as desired.

## 6.2 Basics of induction

We fix a groups $G, H$ and a ground field $k$, and a group morphism $\iota : H \to G$.

### 6.2.1

Corresponding to $\iota$ we have a $k$-algebra morphism $k[H] \to k[G]$ (sending $\sum c_h \cdot \delta_h$ to $\sum c_h \cdot \delta_{\iota(h)}$). By abuse of notation we will denote it by $\iota$ as well. Since we can identify $\mathrm{Rep}_k(G) \cong \mathrm{Mod}(k[G])$ (and similarly for $H$), we already have three functors at our disposal:




Here $\mathrm{res}^G_H := \mathrm{res}^{k[G]}_{k[H]}$, $\mathrm{ind}^H_G$ is the left adjoint of $\mathrm{res}^G_H$ (so can be thought of as $k[G] \underset{k[H]}{\otimes} -$) and $\mathrm{Ind}^H_G$ is the right adjoint of $\mathrm{res}^G_H$ (so can be thought of as $\mathrm{Hom}_{k[H]}(k[G], -)$).

### 6.2.2

Usually one uses the terminology of restriction and induction when $\iota : H \to G$ is an inclusion. Let us assume so in what follows.

### 6.2.3

Let us first get a more concrete feeling of $\mathrm{ind}^H_G$. Let $(g_q)_{q \in G/H}$ be a set of representatives in $G$ of the cosets in $G/H$ (so $g_q \in q$). We then have an isomorphism of right $k[H]$-modules

$$k[G] \cong \oplus_{q \in G/H} k[H]$$

given by sending $(d_q)_{q \in G/H}$ on the right to $\sum_{q \in G/H} \delta_{g_q} \cdot d_q$ on the left. We therefore obtain, given a $k[H]$-module $M$, an isomorphism of $k$-vector spaces

$$k[G] \underset{k[H]}{\otimes} M \cong \left(\oplus_{q \in G/H} k[H]\right) \underset{k[H]}{\otimes} M \cong \oplus_{q \in G/H} \left(k[H] \underset{k[H]}{\otimes} M\right) \cong \oplus_{q \in G/H} M$$

which sends $(m_q)_{q\in G/H}$ on the tight to $\sum_{q\in G/H} \delta_{g_q} \otimes m_q$ on the left.

Therefore, somewhat more concretely, we can think of $\mathrm{ind}_G^H(M)$ as $\oplus_{q\in G/H}$ "$g_q$" $M$ (where we imagine "$g_q$"$m$ to be the result of acting by $g_q$ on $m$), and given $g \in G$, to compute $g($"$g_q$"$m)$ we write $gg_q = g_{q'}h$ for $q' \in G/H$ and $h \in H$ and then $g($"$g_q$"$m) =$ "$g_{q'}$"$(hm)$. So we have actions of $h$'s on $m$'s, and we add actions of $g$'s on $m$'s in the "most economical" way (adding formal action one element in a coset, and then the action of all other elements in that coset is already determined).

In particular, we see that

$$\dim(\mathrm{ind}_G^H(V)) = [G : H] \cdot \dim V.$$

Let us reiterate the structure of the induction. Suppose that we are given an $G$-representation $V$, and an $H$-subrepresentation $W \subset V$. By adjunction, we obtain a $G$-morphism $\mathrm{ind}_G^H W \to V$. Then this is an isomorphism if and only if

$$V = \bigoplus_{q\in G/H} g_q W.$$

This is injective if and only if the subspaces $(g_q W)_{q\in G/H}$ in $V$ are linearly independent (i.e. their sum in $V$ is a direct sum).

### 6.2.4

Let us next get a more concrete feeling of $\mathrm{Ind}_G^H$. Given a $k[H]$-module $M$, we can think of $\mathrm{Hom}_{k[H]}(k[G], M)$ as a subspace of $\mathrm{Hom}_k(k[G], M)$. The latter space we can identify with the space of maps from $G$ to $M$ (since we have a basis of $k[G]$ parametrized by elements of $G$). If we unfold the definitions, we see that the subspace under question is the subspace of functions $f : G \to M$ which satisfy $f(hx) = hf(x)$ for all $h \in H, x \in G$. If we unfold the definitions, we see that the $G$-action on this space is given by $(gf)(x) = f(xg)$. So, to summarize, we can think of $\mathrm{Ind}_G^H(M)$ as being the space

$$\mathrm{Ind}_G^H(M) := \{f : G \to M \mid f(hx) = hf(x) \quad \forall h \in H, x \in G\},$$

and the $G$-action is given by

$$(gf)(x) := f(xg).$$

As an exercise, the reader should write again explicitly, using this last model, the adjunction isomorphism

$$\mathrm{Hom}_H(\mathrm{res}_H^G(N), M) \cong \mathrm{Hom}_G(N, \mathrm{Ind}_G^H(M)).$$

We will always think of $\mathrm{Ind}_G^H$ in this functions-on-$G$ model.

### 6.2.5

These constructions shed light on some previous things. For example, the regular representation $k[G]$ can be interpreted as $\mathrm{ind}_G^1(\mathbb{C})$, and the calculation of how many times an irreducible $G$-representation appears in $k[G]$ can be viewed as using the adjunction of $\mathrm{ind}_G^1$ and $\mathrm{res}_1^G$.

### 6.2.6

We want to compare $\mathrm{ind}_G^H$ and $\mathrm{Ind}_G^H$. We will show that we have morphism of functors $\mathrm{ind}_G^H \to \mathrm{Ind}_G^H$, which is an isomorphism when $[G : H]$ is finite.

By adjunction, to define a morphism of $G$-representations $\mathrm{ind}_G^H(V) \to \mathrm{Ind}_G^H(V)$ is the same as to define a morphism of $H$-representations $V \to \mathrm{Ind}_G^H V$ (here we write $\mathrm{Ind}_G^H V$ instead of $\mathrm{res}_H^G \mathrm{Ind}_G^H V$ by abuse of notation). We define such a morphism by sending $v \in V$ to the function on $G$ sending $h \in H$ to $hv$ and sending $g \in G \smallsetminus H$ to 0. Now, if $[G : H]$ is finite, we define a morphism $\mathrm{Ind}_G^H V \to \mathrm{ind}_G^H V$ as follows. Choose representatives $(g_q)_{q \in H\backslash G}$ in $G$ for the cosets in $H\backslash G$. We send a function $f$ in $\mathrm{Ind}_G^H V$ to $\sum_{q \in H\backslash G} \delta_{g_q^{-1}} \otimes f(g_q)$. We leave to the reader to check that indeed in this way we obtain a morphism inverse to the previous one.

We therefore see that, if $G/H$ is finite (which holds in our main case of interest, when $G$ is finite), there is in some sense no difference between $\mathrm{ind}_G^H$ and $\mathrm{Ind}_G^H$ (one can think that the same functor is both a left adjoint and a right adjoint of the restriction functor, i.e. it so happens that the left and right adjoints of the restriction functor in that case are isomorphic in a specific way).

Let us try a more abstract approach (maybe it can be improved upon). Let us work in the more general setup of a morphism of $k$-algebras $B \to A$ as above. First, we have a morphism of $A$-modules, functorial in the $B$-module $M$:

$$\mathrm{Hom}_B(A, B) \underset{B}{\otimes} M \to \mathrm{Hom}_B(A, M).$$

Here the right $B$-module structure on $\mathrm{Hom}_B(A, B)$ is thanks to the right $B$-module structure on $B$ and the left $A$-module structure on the whole left thing is thanks to the left $A$-module structure on $\mathrm{Hom}_B(A, B)$. Namely, the morphism is characterized by sending $T \otimes m$ on the left to $a \mapsto (T(a)m)$ on the right.

Next, suppose also that we are given a morphism of left $B$-modules $e : A \to B$. We then construct a morphism of left $A$-modules and right $B$-modules

$$A \to \mathrm{Hom}_B(A, B)$$

by sending $a$ to the morphism sending $a'$ to $e(a'a)$. We therefore obtain a morphism of $A$-modules, functorial in the $B$-module $M$:

$$A \underset{B}{\otimes} M \to \mathrm{Hom}_B(A, B) \underset{B}{\otimes} M$$

by applying the last morphism on the left tensor component. Composing, we obtain a morphism of $A$-modules, functorial in the $B$-modules $M$:

$$A \underset{B}{\otimes} M \to \mathrm{Hom}_B(A, B) \underset{B}{\otimes} M \to \mathrm{Hom}_B(A, M).$$

If we want it to be an isomorphism, the easiest way would be to try both morphisms to be isomorphisms separately.

We claim that the second arrow is an isomorphism if $A$ is a free left $B$-module of finite rank. Indeed, we can generalize the morphism, by considering for a left $B$-module $N$ the morphism of $k$-vector spaces, functorial in the $B$-modules $M$:

$$\mathrm{Hom}_B(N, B) \underset{B}{\otimes} M \to \mathrm{Hom}_B(N, M).$$

When $N$ is isomorphic to $B$, we leave to the reader to check that this morphism is an isomorphism. Also, if this morphism is an isomorphism for some $N_1$ and $N_2$, we leave to the reader to check that this morphism as an isomorphism for $N_1 \oplus N_2$. Therefore, if $N$ is a free $B$-module of finite rank, we see that the morphism is an isomorphism. Since a morphism of $A$-modules is an isomorphism if and only if it is an isomorphism of $k$-vector spaces, the claim follows.

The first arrow is an isomorphism if $A \to \mathrm{Hom}_B(A, B)$ is an isomorphism.

In our case we take $A = k[G]$, $B = k[H] \hookrightarrow k[G]$, and $e : k[G] \to k[H]$ sending $\delta_g$ to $\delta_g$ if $g \in H$ and to 0 if $g \notin H$.

## 6.3 Sample application to dimensions of irreducible representations

Throughout this subsection, we fix a finite group $G$ and an algebraically closed ground field $k$ whose characteristic does not divide $|G|$.

### 6.3.1

Let us prove here the following:

**Claim 6.1.** *Let $A \subset G$ be an abelian normal subgroup. Then the dimension of any irreducible $G$-representation over $k$ divides $[G : A]$.*

### 6.3.2

Let us interject with a small discussion around conjugation symmetry, which will be useful in the following.

**Definition 6.2.**

1. Let $\theta : G' \to G$ be an isomorphism of groups. Let $V$ be a $G'$-representation. By ${}^{\theta}V$ we denote the $G$-representation which, as a vector space, is $V$ (but we write "$\theta$"$V$, where "$\theta$" is just a place holder for intuition), and the $G$-action is given by

$$g * \text{“}\theta\text{”} v := \text{“}\theta\text{”} (\theta^{-1}(g) v).$$

2. Let $\theta : G \to G$ be an automorphism. Let $H \subset G$ be a subgroup. Then $\theta$ induces an isomorphism $H \to \theta(H)$ (we denote $\theta$ again by abuse of notation), and therefore given a representation $V$ of $H$, we get a representation ${}^{\theta}V$ of $\theta(H)$.

3. Let $g_0 \in G$. Then we have the automorphism $\theta_{g_0}$ of $G$ given by $\theta_{g_0}(g) := g_0 g g_0^{-1}$. Given a subgroup $H \subset G$ and a representation $V$ of $H$, we denote ${}^{g_0}V := {}^{\theta_{g_0}}V$ (so this is a representation of $g_0 H g_0^{-1}$).

**Exercise 6.1.** *Let $V$ be a $G$-representation and let $g_0 \in G$. Then the $G$-representation ${}^{g_0}V$ is isomorphic to the $G$-representation $V$.*

**Exercise 6.2.** *Let $H \subset G$ be a subgroup and let $\theta$ be an automorphism of $G$. Let $V$ be an $H$-representation. Then $\mathrm{ind}_G^{\theta(H)}({}^{\theta}V) \cong {}^{\theta}(\mathrm{ind}_G^H V)$ (of course, this is part of a general principal, that every natural construction will "transoport" along isomorphisms). Let $g_0 \in G$. Then $\mathrm{ind}_G^{g_0 H g_0^{-1}}({}^{g_0}V) \cong \mathrm{ind}_G^H V$.*

**Exercise 6.3.** *Let $H \subset G$ be a normal subgroup. Let $V$ be a $G$-representation, $F \subset V$ an $H$-isotypic component, and $g_0 \in G$. Then $g_0 F$ is an $H$-isotypic component in $V$ as well: If $F$ was the $E$-isotypic component of $V$, where $E$ is an irreducible representation of $H$, then $g_0 F$ is the ${}^{g_0}E$-isotypic component of $V$.*

### 6.3.3

A piece of terminology: A representation is called **isotypical** if all irreducible representations appearing in it are isomorphic, i.e. if it consists of one isotypic component.

**Lemma 6.3.** *Let $H \subset G$ be a normal subgroup. Let $E$ be an irreducible $G$-representation. Let $F \subset E$ be an $H$-isotypical component. Consider*

$$K = \{g \in G \mid gF = F\} \subset G.$$

*Then $K$ is a subgroup of $G$ containing $H$, and $F$ is a $K$-subrepresentation of $G$. Consider the $G$-morphism $\mathrm{ind}_G^K F \to E$ corresponding to the inclusion $K$-morphism $F \to E$. It is an isomorphism. The $K$-representation $F$ is irreducible.*

*Proof.* I is enough to establish that our morphism is injective, since $E$ is irreducible. As we explained above, for this it is enough to establish that, choosing representatives $(g_q)_{q \in G/K}$ in $G$ for cosets in $G/K$, the subspaces $(g_q F)_{q \in G/K}$ in $E$ are linearly independent. For this, it is enough to show that the subspaces $g_q F$ are all distinct $H$-isotypic components in $E$, since isotypic components are linearly independent. Indeed, recall first that $g_q F$ are $H$-isotypic components, by Exercise 6.3 above. Second, if $g_q F = g_{q'} F$ then $g_{q'}^{-1} g_q F = F$, so $g_{q'}^{-1} g_q \in K$ and so $q' = q$.

We now show that $F$ is an irreducible $K$-representation. If $F$ is not an irreducible $K$-representation, we can write $F = F' \oplus F''$ where $F'$ and $F''$ are non-zero $K$-subrepresentations. Then $E \cong \mathrm{ind}_G^K(F) \cong \mathrm{ind}_G^K(F') \oplus \mathrm{ind}_G^K(F'')$, contradicting $E$ being irreducible (note to self: better to not assume semisimplicity where not needed, so better here to say that if $F$ is not irreducible then we take a non-trivial surjection $F \to F'$, and use $\mathrm{ind}_G^K$ being right exact..). □

**Lemma 6.4.** *Let $H \subset G$ be a normal subgroup. Let $E$ be an irreducible $G$-representation. Then either $\mathrm{res}_H^G(E)$ is isotypical or there exists a subgroup $H \subset K \subset G$ such that $K \neq G$ and $E$ is isomorphic to a representation of the form $\mathrm{ind}_G^K(F)$ where $F$ is an irreducible representation of $K$.*

*Proof.* Let $F \subset \mathrm{res}_H^G(E)$ be an isotypic component. Let us consider $K := \{g \in G \mid gF = F\}$. Then $H \subset K \subset G$ and $F$ is a $K$-subrepresentation of $E$. By the previous lemma, the $K$-representation $F$ is irreducible and the inclusion of $K$-representations $F \to E$ gives rise, by adjunction, to a morphism of $G$-representations $\mathrm{ind}_G^K(F) \to E$ which is an isomorphism. We just need to see that if $K = G$ then $\mathrm{res}_H^G(E)$ is isotypical. This is clear, since if $K = G$ then $F$ is a $G$-subrepresentation of $E$ and therefore, sicne $E$ is irreducible, we must have $F = E$, i.e. $E$ is isotypical as an $H$-representation.

□

### 6.3.4

*Proof (of Claim 6.1).* We proceed by induction on $|G|$. Applying Lemma 6.4 to our situation (where $H$ of the lemma is our $A$) we consider two cases. In the first, $\mathrm{res}_A^G(E)$ is isotypical. Since $A$ is abelian, an isotypical $A$-representation is in fact a representation on which $A$ acts by scalars. Denoting by $\rho : G \to GL(E)$ the action map, we have therefore that $\rho(A)$ lies in the center of $\rho(G)$. We can consider $E$ as a representation of $\rho(G)$, still irreducible obviously. By Proposition 5.33 we see that $\dim E$ divides $[\rho(G) : \rho(A)]$. Since $[\rho(G) : \rho(A)]$ divides $[G : A]$ we are done in this case. In the second case, $E$ is isomorphic to a representation $\mathrm{ind}_G^K(F)$ where $A \subset K \subset G$ with $K \neq G$ and $F$ is an irreducible $K$-representation. By induction, we already know that $\dim F$ divides $[K : A]$. Therefore

$$\dim E = \dim(\mathrm{ind}_G^K(F)) = [G : K] \cdot \dim F \quad \mid \quad [G : K] \cdot [K : A] = [G : A].$$

□

## 6.4 Another application - irreducible representations of semidirect products $A \rtimes H$, with $A$ commutative

Throughout this subsection, we let $G = A \rtimes H$, where $A$ is a finite abelian group and $H$ a finite group, and $k$ an algebraically closed ground field whose characteristic does not divide $|G|$.

### 6.4.1

The group $H$ acts on the group $A$ by group automorphisms, and therefore it also acts on the group $\mathrm{Ch}_k(A)$ of characters of $A$.

### 6.4.2

For a $G$-representation $V$ and for $\chi \in \mathrm{Ch}_k(A)$, we denote

$$V_{A,\chi} = \{v \in V \mid av = \chi(a)v \ \forall a \in A\} \subset V$$

(this is the isotypical component of $\mathrm{res}^G_A V$ corresponding to the irreducible $A$-representation $k_\chi$). Let us also denote by $\mathrm{supp}(V) \subset \mathrm{Ch}_k(A)$ the subset consisting of $\chi$ for which $V_{A,\chi} \neq 0$. Notice that $\mathrm{supp}(V)$ is a $G$-invariant subset of $\mathrm{Ch}_k(A)$, and if $V$ is finite-dimensional and non-zero then $\mathrm{supp}(V) \neq \emptyset$. Further, for $\chi \in \mathrm{Ch}_k(A)$ we denote

$$G_\chi := \{g \in G \mid g\chi = \chi\} \subset G$$

and we have $G_\chi = A \rtimes H_\chi$ where

$$H_\chi := \{h \in H \mid h\chi = \chi\} \subset H.$$

Since $gV_{A,\chi} = V_{A,g\chi}$, we see that if $V_{A,\chi} \neq 0$ then

$$\{g \in G \mid gV_{A,\chi} = V_{A,\chi}\} = G_\chi.$$

### 6.4.3

**Lemma 6.5.** *Suppose that $V$ is finite-dimensional and non-zero. Then $V$ is $G$-irreducible if and only if $\mathrm{supp}(V)$ is $G$-transitive (i.e. consists of precisely one $G$-orbit in $\mathrm{Ch}_k(A)$) and, for one/all $\chi \in \mathrm{supp}(V)$, $V_{A,\chi}$ is $H_\chi$-irreducible. In that case, the inclusion $G_\chi$-morphism $V_{A,\chi} \to V$ induces a $G$-isomorphism $\mathrm{ind}^{G_\chi}_G(V_{A,\chi}) \xrightarrow{\sim} V$.*

*Proof.* Suppose that $V$ is irreducible. Let $\chi \in \mathrm{supp}(V)$. Since $V' := \sum_{g\in G} gV_{A,\chi} = \sum_{g\in G} V_{A,g\chi}$ is a non-zero $G$-subrepresentation of $V$, we must have $V' = V$. Thus $\mathrm{supp}(V) = G\chi$, i.e. a single $G$-orbit. Furthermore, Lemma 6.3 shows that $V_{A,\chi}$ is an irreducible $G_\chi$-representation (and hence an irreducible $H_\chi$ representation, since $A$ acts on $V_{A,\chi}$ by scalars, and so $G_\chi$-subrepresentations of $V_{A,\chi}$ are the same as $H_\chi$-subrepresentations). Furthermore, that Lemma states that the inclusion $G_\chi$-morphism $V_{A,\chi} \to V$ induces a $G$-isomorphism $\mathrm{ind}^{G_\chi}_G(V_{A,\chi}) \xrightarrow{\sim} V$, as desired.

Conversely, suppose that $\mathrm{supp}(V)$ is a single $G$-orbit, say $G\chi$, and that $V_{A,\chi}$ is $H_\chi$-irreducible. Let $V' \subset V$ be a non-zero $G$-subrepresentation. Then $\mathrm{supp}(V') \neq \emptyset$. Since $\mathrm{supp}(V') \subset \mathrm{supp}(V)$ and it is $G$-invariant, we must have $\mathrm{supp}(V') = \mathrm{supp}(V)$. So $\chi \in \mathrm{supp}(V')$. Hence $V' \cap V_{A,\chi} \neq 0$. But $V' \cap V_{A,\chi}$ is a $H_\chi$-subrepresentation of $V_{A,\chi}$ and therefore, since $V_{A,\chi}$ is $H_\chi$-irreducible, we must have $V' \cap V_{A,\chi} = V_{A,\chi}$, i.e. $V_{A,\chi} \subset V'$. But then, for every $g \in G$, we have $V_{A,g\chi} = gV_{A,\chi} \subset gV' = V'$. Therefore $V = \sum_{g\in G} V_{A,g\chi} \subset V'$ and so $V' = V$, showing that $V$ is indeed $G$-irreducible. $\square$

### 6.4.4

Given $\chi \in \mathrm{Ch}_k(A)$ and a $H_\chi$-representation $F$, let us consider the $G_\chi$-representation $F_\chi$, which is the same vector space as $F$, with the unique action of $G_\chi$ such that $H_\chi$ acts as before while $A$ acts by $\chi$ (check that it is indeed well-defined). From the above Lemma we see that every irreducible $G$-representation is isomorphic to $\mathrm{ind}_G^{G_\chi} F_\chi$ for some $\chi \in \mathrm{Ch}_k(A)$ and $H_\chi$-irrep. $F$. Conversely, let $\chi \in \mathrm{Ch}_k(A)$ and let $F$ be a $H_\chi$-irrep. We want to see that $V := \mathrm{ind}_G^{G_\chi} F_\chi$ is $G$-irreducible. Let $(h_q)_{q \in G/G_\chi}$ be a set of representatives in $H$ for cosets in $G/G_\chi$. Recall that

$$V = \bigoplus_q h_q F_\chi.$$

Notice that $F_\chi \subset V_{A,\chi}$, but in fact we cliam that we have an equality. Indeed, notice that for $q \neq G_\chi$ we have $h_q F_\chi \subset V_{A, h_q \chi}$ and $h_q \chi \neq \chi$, and from this the claim follows. Thus we readily see that $\mathrm{supp}(V) = G\chi$ and that $V_{A,\chi}$ is an irreducible $G_\chi$-representation (and so an irreducible $H_\chi$-representation).

### 6.4.5

It is now left to account for coincidences in the above recipe.

Let $\chi \in \mathrm{Ch}_k(A)$ and $F$ an irreducible $H_\chi$-representation. Let $h \in H$. Then we have

$$\mathrm{ind}_G^{G_\chi}(F_\chi) \cong \mathrm{ind}_G^{hG_\chi h^{-1}}({}^h(F_\chi)) = \mathrm{ind}_G^{G_{h\chi}}(({}^hF)_{h\chi}).$$

This shows that in our recipe, it is enough, when we run over $\chi$'s, to only consider one $\chi$ in each $H$-orbit on $\mathrm{Ch}_k(A)$.

We now claim that different $H$-orbits in $\mathrm{Ch}_k(A)$ give different things, i.e. that if $\mathrm{ind}_G^{G_\chi} F_\chi \cong \mathrm{ind}_G^{G_{\chi'}} F'_{\chi'}$ (in our usual notation here), then $\chi'$ lies in the same $H$-orbit as $\chi$. Indeed, from the analysis above we see that $G_\chi$ is recoverable as $\mathrm{supp}(\mathrm{ind}_G^{G_\chi}(F_\chi))$.

Now, let $F'$ be another irreducible $H_\chi$-representation, and assume $\mathrm{ind}_G^{G_\chi} F_\chi \cong \mathrm{ind}_G^{G_\chi} F'_\chi$. Then $F'$ can be recovered as the $(A,\chi)$-isotypic component on the right, and so on the left, but $F$ is also the $(A,\chi)$-isotypic component on the left. So $F'$ and $F$ are isomorphic (as $G_\chi$-representations, and in particular $H_\chi$-representations).

### 6.4.6

To summarize, we can parametrize irreducible $G$-representations as follows. We run over $\chi$ in $\mathrm{Ch}_k(A)$, but only one in each $H$-orbit. We then run over all irreducible $H_\chi$-representations $F$ (up to isomorphism). We consider $\mathrm{ind}_G^{G_\chi} F$. This gives all irreducible $G$-representations (up to isomorphism), without repetitions.

#### 6.4.7

Let us give an example. Let $\mathbb{F}$ be a finite field with $q$ elements and consider $G := \mathbb{F} \rtimes \mathbb{F}^\times$, where $\mathbb{F}^\times$ acts on $\mathbb{F}$ via multiplication (this is isomorphic to the group of affine linear transformations of a one-dimensional vector space over $\mathbb{F}$). We will work over $\mathbb{C}$. Given $\psi \in \mathrm{Ch}_{\mathbb{C}}(\mathbb{F})$ and $c \in \mathbb{F}$, we can construct the character $c\psi$ as sending $x$ to $\psi(cx)$. It is known that this gives $\mathrm{Ch}_{\mathbb{C}}(\mathbb{F})$ the structure of a 1-dimensional vector space over $\mathbb{F}$. In other words, if we fix $\psi_0 \neq 1$, then $c \mapsto c\psi_0$ is a bijection $\mathbb{F} \to \mathrm{Ch}_{\mathbb{C}}(\mathbb{F})$. Recall that $\mathbb{F}^\times$ acts on our $\mathbb{F}$ by multiplication, and so on $\mathrm{Ch}_{\mathbb{C}}(\mathbb{F})$ it acts by $c * \psi = c^{-1}\psi$. We have two orbits for that action - the singleton consisting of the trivial characters, and its complement. So representatives for the orbits are 1 and $\psi_0$. The stabilizer in $\mathbb{F}^\times$ of 1 is the whole $\mathbb{F}^\times$, while the stabilizer of $\psi_0$ is the trivial subgroup. We therefore obtain the following parametrization of irreducible representations of $G$. For every character $\chi : \mathbb{F}^\times \to \mathbb{C}^\times$ we have the character $G \to \mathbb{F}^\times \xrightarrow{\chi} \mathbb{C}^\times$ (where the first map is the standard projection), providing 1-dimensional irreducible representations. In addition, we have the $(q-1)$-dimensional irreducible representation $\mathrm{ind}_G^{\mathbb{F}}\mathbb{C}_{\psi_0}$. Quick check that (some of) the numerics is OK: $(q-1) \cdot 1^2 + (q-1)^2 = q(q-1) = |G|$.

## 6.5 Characters of induced representations

Throughout this subsection, we fix a finite group $G$ and an algebraically closed ground field $k$, whose characteristic does not divide $|G|$ (one can do things a bit more generally, but let us not bother). We also fix a subgroup $H \subset G$.

#### 6.5.1

We would like to figure out a linear map

$$^{\mathrm{fun}}\mathrm{Ind}_G^H : \mathrm{Fun}_k(H)^{cl} \to \mathrm{Fun}_k(G)^{cl}$$

such that, for a finite-dimensional $H$-representation $V$, we will have

$$^{\mathrm{fun}}\mathrm{Ind}_G^H(\mathrm{ch}_V) = \mathrm{ch}_{\mathrm{Ind}_G^H V}.$$

In fact, it is clear that such a map exists and is unique, because the equality holds for all finite-dimensional $V$ if and only if it holds for all irreducible $V$, and the vectors $\mathrm{ch}_V$, as $V$ runs over irreducible $H$-representations form a basis for $\mathrm{Fun}_k(H)^{cl}$.

#### 6.5.2

Notice that it is clear that for a finite-dimensional $G$-representation $W$ we have

$$\mathrm{ch}_{\mathrm{res}_H^G W} = (\mathrm{ch}_W)|_H.$$

(simply the restriction of a function). Recall that we saw that the symmetric bilinear form $\langle -, - \rangle$ on $\mathrm{Fun}_k(G)^{cl}$ is non-degenerate (and similarly for $H$). Since characters span the spaces of class functions, the equality

$$\langle \mathrm{ch}_W, {}^{\mathrm{fun}}Ind_G^H(\mathrm{ch}_V)\rangle = \langle \mathrm{ch}_W, \mathrm{ch}_{\mathrm{Ind}_G^H V}\rangle = \dim \mathrm{Hom}_G(W, \mathrm{Ind}_G^H V) =$$

$$= \dim \mathrm{Hom}_H(\mathrm{res}_H^G W, V) = \langle \mathrm{ch}_{\mathrm{res}_H^G W}, \mathrm{ch}_V\rangle = \langle (\mathrm{ch}_W)|_H, \mathrm{ch}_V\rangle$$

shows that we can characterize ${}^{\mathrm{fun}}\mathrm{Ind}_G^H$ as the operator which is adjoint to the restriction operator $(-)|_H$ with respect to the bilinear forms we have for $G$ and $H$.

### 6.5.3

Let us now calculate the map ${}^{\mathrm{fun}}\mathrm{Ind}_G^H$ more explicitly. Notice that we can think of our restriction map $(-)|_H : \mathrm{Fun}_k(G)^{cl} \to \mathrm{Fun}_k(H)^{cl}$ as the composition of three maps

$$\mathrm{Fun}_k(G)^{cl} \to \mathrm{Fun}_k(G) \to \mathrm{Fun}_k(H) \to \mathrm{Fun}_k(H)^{cl}$$

where the first map is the inclusion, the second map is the restriction map, and the third map is the $H$-conjugation averaging we have already encountered before. Recall that also saw before that the conjugation averaging is adjoint to the inclusion (with respect to our bilinear forms). We therefore see that our ${}^{\mathrm{fun}}\mathrm{Ind}_G^H(\chi)$ is the composition

$$\mathrm{Fun}_k(H)^{cl} \to \mathrm{Fun}_k(H) \to \mathrm{Fun}_k(G) \to \mathrm{Fun}_k(G)^{cl},$$

were the first map is the inclusion, the second map is the adjoint to the restriction on all functions (not only class functions) and the third map is the $G$-conjugation averaging. It is elementary (i.e. left as an exercise) to see that the second map sends a function $f \in \mathrm{Fun}_k(H)$ to the function on $G$ which sends $x$ to $\frac{|G|}{|H|} f(x)$ if $x \in H$ and to 0 otherwise. Therefore, we can calculate that given $f \in \mathrm{Fun}_k(H)^{cl}$ we have:

$${}^{\mathrm{fun}}\mathrm{Ind}_G^H(f)(x) = \frac{1}{|G|} \sum_{g \in G \text{ s.t. } g^{-1}xg \in H} \frac{|G|}{|H|} \cdot f(g^{-1}xg) = \frac{1}{|H|} \sum_{g \in G \text{ s.t. } g^{-1}xg \in H} f(g^{-1}xg).$$

Notice that if $g \in G$ appears in the sum, i.e. $g^{-1}xg \in H$, then $gh$ also appears for all $h \in H$, and the value is the same, as $f$ is a class function: $f((gh)^{-1}x(gh)) = f(h^{-1}(g^{-1}xg)h) = f(g^{-1}xg)$. Therefore we can adjust for this redundancy and finally write:

$${}^{\mathrm{fun}}\mathrm{Ind}_G^H(f)(x) = \sum_{g \in G/H \text{ s.t. } g^{-1}xg \in H} f(g^{-1}xg).$$

Here the meaning is that we choose a representative $g \in G$ for an element in $G/H$, and everything that appears in the formula (i.e. the condition $g^{-1}xg \in H$ as well as the value $f(g^{-1}xg)$) will not depend on this choice of representative.

### 6.5.4

Let us re-interpret the formula as follows. We consider the $G$-set $Y := G/H$. Denote $y_0 = H \in Y$. Let again $f \in \mathrm{Fun}_k(H)^{cl}$. Let us consider the "stabilizers" set

$$S_Y := \{(x, y) \in G \times Y \mid xy = y\}.$$

Using $f$ we obtain a function $\widetilde{f}$ on $S_Y$ as follows. Let $(x, y) \in S_Y$. Since the action of $G$ on $Y$ is transitive, there exists $t \in G$ such that $ty_0 = y$. Then $t^{-1}xt$ stabilizes $y_0$, and so lies in $H$. We set $\widetilde{f}((x, y)) := f(t^{-1}xt)$. One immediately checks that, since $f$ is a class function, the function $\widetilde{f}$ is well-defined (i.e. there was independence on the choice of $t$). We now reformulate (it is an easy exercise to verify that this indeed is a reformulation):

$$^{\mathrm{fun}}\mathrm{Ind}_G^H(f)(x) = \sum_{y \in Y \text{ s.t. } xy=y} \widetilde{f}(x, y).$$

Thus, the formula for the character of induction is a "fixed point formula".

### 6.5.5

Let us compute the character of the $G := \mathbb{F} \rtimes \mathbb{F}^\times$-representation $E := \mathrm{ind}_G^{\mathbb{F}} \mathbb{C}_{\psi_0}$ from the example in §6.4.7. Let $g := (x, t) \in \mathbb{F} \rtimes \mathbb{F}^\times$. Since $\{(0, s) : \ s \in \mathbb{F}^\times\}$ form representatives for $G/\mathbb{F}$, and since for $s \in \mathbb{F}^\times$ we have

$$(0, s)(x, t)(0, s^{-1}) = (sx, t),$$

we obtain that $\mathrm{ch}_E(x, t) = 0$ if $t \neq 1$ and $\mathrm{ch}_E(x, t) = \sum_{s \in \mathbb{F}^\times} \psi_0(sx)$ if $t = 1$, and so $\mathrm{ch}_E(0, 1) = q - 1$ and $\mathrm{ch}_E(x, 1) = -1$ for $x \neq 0$.

## 6.6 Mackey's theory

Throughout this subsection we fix a finite group $G$ and an algebraically closed ground field $k$ whose characteristic does not divide $|G|$.

### 6.6.1

The basic motivation is that given a subgroup $H \subset G$, we would like to obtain irreducible representations of $G$ by inducing irreducible representations of $H$. Notice first that every irreducible representation $F$ of $G$ appears as a sub-representation of $\mathrm{Ind}_G^H E$, for some irreducible representation $E$ of $H$. Indeed, as $\mathrm{res}_H^G F \neq 0$, there exists an irreducible representation $E$ of $H$ and a surjection $\mathrm{res}_H^G F \to E$ (we take the projection on a irreducible summand in $\mathrm{res}_H^G F$). By applying adjunction, we obtain a non-zero map $F \to \mathrm{Ind}_G^H E$. Since $F$ is irreducible, the non-zero map must be an injection.

Therefore, we get a strategy for how to construct irreducible representations of $G$, given that we know irreducible representations of $H$: Consider the inductions, study how they decompose into irreducibles, and what coincidences with get in this process.

The problem is that, if we take $H = G$ then we clearly get nothing interesting. If we take $H = 1$ then we get that every irreducible representation of $G$ appears in the regular representation, which we already exploited. The problem with the latter is that $\mathrm{Ind}_G^H E$ can be thought of as combining the information of the $H$-representation $E$ and the structure of the $G$-set $G/H$, and if $G/H$ is too big, its structure will be a ("geometric") problem on its own, and we will have trouble parametrizing into which irreducibles does $\mathrm{Ind}_G^H E$ decompose. For example, $\mathrm{Ind}_G^H E$ will have almost no chance of being irreducible. So the point is to look for $H$ for which $G/H$ is relatively small, so that we can expect to understand how to decompose $\mathrm{Ind}_G^H E$ into irreducibles.

So a basic motivating question would be, for example, given an irreducible $H$-representation $E$, is $\mathrm{Ind}_G^H(E)$ an irreducible representation of $G$? Notice that a finite dimensional representation $V$ of $G$ is irreducible if and only if $\dim \mathrm{Hom}_G(V, V) = 1$. So we want to calculate

$$\dim \mathrm{Hom}_G(\mathrm{Ind}_G^H E, \mathrm{Ind}_G^H E) = \dim \mathrm{Hom}_H(\mathrm{res}_H^G \mathrm{Ind}_G^H E, E).$$

Therefore we want to gain an understanding of $\mathrm{res}_H^G \mathrm{Ind}_G^H E$, and it turns out to make sense to investigate the more general expression $\mathrm{res}_K^G \mathrm{Ind}_G^H E$, where $K \subset G$ is another subgroup.

### 6.6.2

**Proposition 6.6.** *Let $H, K \subset G$. Choose representatives $(g_q)_{q \in H\backslash G/K}$ in $G$ of the double cosets in $H\backslash G/K$ (so $g_q \in q$). We will construct an isomorphism of functors*

$$\mathrm{res}_K^G \circ \mathrm{Ind}_G^H \cong \bigoplus_{q \in H\backslash G/K} \mathrm{Ind}_K^{K \cap g_q^{-1} H g_q} \circ \mathrm{res}_{K \cap g_q^{-1} H g_q}^{g_q^{-1} H g_q} \circ {}^{g_q^{-1}}(-).$$

*In other words, for every $H$-representation $V$ we construct an isomorphism*

$$\mathrm{res}_K^G \mathrm{Ind}_G^H(V) = \bigoplus_{q \in H\backslash G/K} S_q(V) \cong \bigoplus_{q \in H\backslash G/K} \mathrm{Ind}_K^{K \cap g_q^{-1} H g_q}(\mathrm{res}_{K \cap g_q^{-1} H g_q}^{g_q^{-1} H g_q} {}^{g_q^{-1}}V),$$

*and it is functorial in $V$.*

*Proof.* Recall that $\mathrm{res}_K^G \mathrm{Ind}_G^H(V)$ consists of functions $f : G \to V$ that satisfy $f(hx) = hf(x)$ for all $x \in G$ and $h \in H$. The action of $K$ on it is $(kf)(x) = f(xk)$.

Given $q \in H\backslash G/K$, let us consider the subspace $S_q(V) \subset \mathrm{res}_K^G \mathrm{Ind}_G^H(V)$ consisting of those $f$ for which $f(x) = 0$ if $x \notin q$. It is clear that $S_q(V)$ is a $K$-subrepresentation of $\mathrm{res}_K^G \mathrm{Ind}_G^H(V)$. Also, it is clear that

$$\mathrm{res}_K^G \mathrm{Ind}_G^H(V) = \bigoplus_{q \in H\backslash G/K} S_q(V).$$

We want to understand more the $K$-representations $S_q(V)$. Clearly, $S_q(V)$ is isomorphic to the space of functions $f : q \to V$ which satisfy $f(hx) = hf(x)$ for $x \in q$ and $h \in H$. Given $f \in S_q(V)$, we consider the function $f' : K \to V$ given by $f'(k) := f(g_q k)$. We obtain a linear map

$$S_q(V) \to \mathrm{Fun}_k(K) : \ f \mapsto f'.$$

Notice that for $k \in K$ and $\ell \in K \cap g_q^{-1} H g_q$ we have:

$$f'(\ell k) = f(g_q \ell k) = f(g_q \ell g_q^{-1} \cdot g_q k) = (g_q \ell g_q^{-1}) f(g_q k) = (g_q \ell g_q^{-1}) f'(k).$$

Thus,

$$f' \in \mathrm{Ind}_K^{K \cap g_q^{-1} H g_q} (\mathrm{res}^{g_q^{-1} H g_q}_{K \cap g_q^{-1} H g_q} {}^{g_q^{-1}} V) \subset \mathrm{Fun}_k(K).$$

We leave to the reader to check immediately that our thus obtained

$$\Phi_q : S_q(V) \to \mathrm{Ind}_K^{K \cap g_q^{-1} H g_q} (\mathrm{res}^{g_q^{-1} H g_q}_{K \cap g_q^{-1} H g_q} {}^{g_q^{-1}} V)$$

is a $K$-representation morphism. We claim that $\Phi_q$ is an isomorphism. To describe the inverse, given $f'$ we construct $f$ as follows. Given $g \in q$, we write $g = h g_q k$ for some $h \in H$ and $k \in K$ and set $f(g)$ to be $hf'(k)$. We now leave to the reader verify that indeed $\Phi_q$ is an isomorphism.

To conclude, we obtained an isomorphism

$$\mathrm{res}^G_K \mathrm{Ind}^H_G(V) = \bigoplus_{q \in H \backslash G / K} S_q(V) \cong \bigoplus_{q \in H \backslash G / K} \mathrm{Ind}_K^{K \cap g_q^{-1} H g_q} (\mathrm{res}^{g_q^{-1} H g_q}_{K \cap g_q^{-1} H g_q} {}^{g_q^{-1}} V).$$

One immediately sees that it is functorial in $V$. □

### 6.6.3

Given a group $K$, let us denote by $\mathrm{Irr}_k(K)$ the set of isomorphism classes of irreducible $K$-representations over $k$. Given an irreducible $K$-representation $E$, we will denote by $[E] \in \mathrm{Irr}_k(K)$ the corresponding isomorphism class.

Let $H \subset G$ be a normal subgroup. We have an action of $G$ on $\mathrm{Irr}_k(H)$, where the result of applying an element $g \in G$ to the isomorphism class of an irreducible $H$-representation $E$ is the isomorphism class of the irreducible $H$-representation ${}^g E$. Notice that $H$ acts trivially here, since ${}^h E$ is isomorphic to $E$ (an exercise). Therefore we obtain an action of $G/H$ on $\mathrm{Irr}_k(H)$.

**Proposition 6.7.** *Let $H \subset G$ be a normal subgroup. Let $E, F$ be irreducible $H$-representations.*

1. *The dimension of $\mathrm{Hom}_G(\mathrm{Ind}^H_G E, \mathrm{Ind}^H_G F)$ is equal to the cardinality of*

$$\mathrm{Trans}_{G/H}([E],[F]) := \{\gamma \in G/H \mid {}^\gamma [E] = [F]\}.$$

2. $\mathrm{Ind}_G^H E$ *is irreducible if and only if* $[E]$ *is a free point for the* $G/H$*-action (i.e.* $\mathrm{Stab}_{G/H}([E]) := \mathrm{Trans}_{G/H}([E],[E]) = \{1\}$.

3. $\mathrm{Ind}_G^H E$ *and* $\mathrm{Ind}_G^H F$ *have common irreducible constituents if and only if* $[E]$ *and* $[F]$ *lie in the same* $G/H$*-orbit. In fact, in that case* $\mathrm{Ind}_G^H E$ *and* $\mathrm{Ind}_G^H F$ *are isomorphic.*

*Proof.* We choose representatives $(g_q)_{q \in G/H}$ in $G$ for cosets in $G/H$ (which are the same as double cosets in $H\backslash G/H$, since $H$ is normal in $G$).

We will use Mackey's theory for $K = H$. We have:

$$\mathrm{Hom}_G(\mathrm{Ind}_G^H E, \mathrm{Ind}_G^H F) \cong \mathrm{Hom}_H(\mathrm{res}_H^G \mathrm{Ind}_G^H E, F) \cong \bigoplus_{q \in G/H} \mathrm{Hom}_H({}^{g_q^{-1}}E, F).$$

Notice that each summand on the right is of dimension 1 or 0, according to whether ${}^{g_q^{-1}}E$ and $F$ are isomorphic or not. This shows the desired dimension assertion.

The second assertion follows by recalling that a $G$-representation $V$ is irreducible if and only if the dimension of $\mathrm{End}_G(V)$ is 1. The first clause of the third assertion follows by recalling that $G$-representations $V$ and $W$ have some common irreducible constituent if and only if $\mathrm{Hom}_G(V, W) \neq 0$. It remains to be seen that if $[E]$ and $[F]$ lie in the same $G/H$-orbit then in fact $\mathrm{Ind}_G^H E$ and $\mathrm{Ind}_G^H F$ are isomorphic. Let $g \in G$ be such that ${}^g E$ is isomorphic to $F$. So $\mathrm{Ind}_G^H F$ is isomorphic to $\mathrm{Ind}_G^H({}^g E)$, which is isomorphic to ${}^g(\mathrm{Ind}_G^H E)$ by an exercise we had, which is isomorphic to $\mathrm{Ind}_G^H E$ by an exercise we had. Combining the chain of isomorphisms, we obtain that $\mathrm{Ind}_G^H F$ is isomorphic to $\mathrm{Ind}_G^H E$. □

### 6.6.4

Let us assume, as an example, that $H \subset G$ is a normal subgroup of index 2. Then by the above theory, we see that we can perform the classification of irreducible $G$-representations, in terms of irreducible $H$-representations, as follows. We run over orbits of the $G/H$-action on $\mathrm{Irr}_k(H)$; Each consists of a single point or two points. If an orbit is a single point $[E]$, then $\mathrm{Ind}_G^H E$ breaks down as the sum of two non-isomorphic irreducible representations. If an orbit consists of two points, one of them $[E]$, then $\mathrm{Ind}_G^H E$ is an irreducible representation (taking the second point $[E']$ in the same $G/H$-orbit will give us $\mathrm{Ind}_G^H E'$ which is isomorphic to $\mathrm{Ind}_G^H E$). In this way we obtain all irreducible $G$-representations, without repetitions.

### 6.6.5

Let us take, as an example, the dihedral group $G := \langle r \rangle \rtimes \langle s \rangle$, where $r$ is an element of order $n$, $s$ is an element of order 2, and $srs^{-1} = r^{-1}$. We have the normal subgroup $H := \langle r \rangle \subset G$ of index 2. Let us work over $\mathbb{C}$. We can identify $\mathrm{Irr}_{\mathbb{C}}(H) \cong \mu_n$ (here $\mu_n \subset \mathbb{C}^\times$ is the subgroup of roots of unity of order dividing

$n$), by first identifying $\mathrm{Irr}_{\mathbb{C}}(H)$ with the group of characters of $H$ (since $H$ is abelian), and the group of characters identifying with $\mu_n$ by sending a character $\chi : H \to \mathbb{C}^\times$ to $\chi(r)$. To see how $G/H$ acts on $\mu_n \cong \mathrm{Irr}_{\mathbb{C}}(H)$ we need to see how $s$ acts. Given a character $\chi : H \to \mathbb{C}^\times$, we have the corresponding $[\mathbb{C}_\chi] \in \mathrm{Irr}_{\mathbb{C}}(H)$. We compute that for $c \in {}^s\mathbb{C}_\chi$ we have

$$r * (\text{“}s\text{”}c) = \text{“}s\text{”}((s^{-1}rs)c) = \text{“}s\text{”}(\chi(s^{-1}rs)c) = \text{“}s\text{”}(\chi(r)^{-1}c).$$

So ${}^s\mathbb{C}_\chi \cong \mathbb{C}_{\chi^{-1}}$. So $s$ acts on $\mu_n \cong \mathrm{Irr}_{\mathbb{C}}(H)$ by $\alpha \mapsto \alpha^{-1}$. Therefore, given $\alpha \in \mu_n$ such that $\alpha \notin \{1, -1\}$, we obtain an irreducible $G$-representation of dimension 2 - $\mathrm{Ind}_G^H \mathbb{C}_{\chi_\alpha}$, where $\chi_\alpha : H \to \mathbb{C}^\times$ is the character sending $r$ to $\alpha$. If $\alpha \in \{1, -1\}$, the representation $\mathrm{Ind}_G^H \mathbb{C}_{\chi_\alpha}$ decomposes into two non-isomorphic irreducible representations, which must be of dimension 1. So those will simply correspond to characters of $G$ (one can write them explicitly). In particular, we see that $G$ has $n+1$ irreducible representations if $n$ is odd and $n+2$ irreducible representations if $n$ is even. Of course, this will also be the number of conjugacy classes in $G$.

It is not hard to write a character table for $G$ now (we use the abbreviation “$n$-dimensional character” for “character of an $n$-dimensional irreducible representation):

- We have the trivial 1-dimensional character, sending $r^i$ to 1 and $sr^i$ to 1, and we have the second 1-dimensional character whose restriction to $H$ is trivial, sending $r^i$ to 1 and $sr^i$ to $-1$.
- If $n$ is even, we have two additional multiplicative characters: One sending $r^i$ to $(-1)^i$ and $sr^i$ to $(-1)^i$ and the other sending $r^i$ to $(-1)^i$ and $sr^i$ to $(-1)^{i+1}$.
- For every $\alpha \in \mu_n \smallsetminus \{1, -1\}$ we have a 2-dimensional character, and those are the same for $\alpha$ and $\alpha^-1$, but otherwise there are no repetitions. We use the formula we had above for the character of an induced representation, to calculate our character. We see that our character sends $r^i$ to $\alpha^i + \alpha^{-i}$ and sends $sr^i$ to 0.

To write a nice character table, it might be a good idea to further write explicitly the conjugacy classes in $G$.

### 6.6.6

We want to briefly discuss symmetries of functions on a line - a line over a finite field since we are working with finite groups. So fix a finite field $\mathbb{F}$. It is convenient to fix a non-trivial character $\psi \in \mathrm{Ch}_{\mathbb{C}}(\mathbb{F})$ (and then, as we have already mentioned, the map $\mathbb{F} \to \mathrm{Ch}_{\mathbb{C}}(\mathbb{F})$ given by $a \mapsto (x \mapsto \psi(ax))$ is a bijection). We have the following two straightforward actions of $\mathbb{F}$ on $\mathrm{Fun}_{\mathbb{C}}(\mathbb{F})$ (here $a \in \mathbb{F}, f \in \mathrm{Fun}_{\mathbb{C}}(\mathbb{F})$):

$$(a *_1 f)(x) := f(x + a),$$

$$(a *_2 f)(x) := \psi(ax)f(x).$$

Let us compute their commutation relation, given $a, b \in \mathbb{F}$:

$$(b *_2 (a *_1 f))(x) = \psi(bx)f(x+a),$$

$$(a *_1 (b *_2 f))(x) = \psi(ba) \cdot \psi(bx)f(x+a).$$

Since the two actions don't commute, we don't obtain an action of $\mathbb{F} \times \mathbb{F}$., To fix this, we need to add the error terms $\psi(-ba)$ to the group, so to speak. Namely, we add a third action of $\mathbb{F}$ on $\mathrm{Fun}_{\mathbb{C}}(\mathbb{F})$:

$$(a *_3 f)(x) := \psi(a)f(x).$$

We now let $(c, b, a) \in \mathbb{F}^3$ act by $(c, b, a)f := c*_3(b*_2(a*_1 f))$. Then (abbreviating the notation - hope the reader will understand)

$$(c_2, b_2, a_2)((c_1, b_1, a_1)f) = c_2 b_2 a_2 c_1 b_1 a_1 f = c_2 c_1 b_2 (a_2 b_1) a_1 f =$$

$$= (b_1 a_2) *_3 c_2 c_1 b_2 b_1 a_2 a_1 f = (c_2 + c_1 + b_1 a_2, b_2 + b_1, a_2 + a_1)f.$$

We are therefore led to consider the **Heisenberg group** $H$, which as a set is $\mathbb{F}^3$, with the product

$$(c_2, b_2, a_2) \cdot (c_1, b_1, a_1) := (c_2 + c_1 + b_1 a_2, b_2 + b_1, a_2 + a_1).$$

Incidentally, notice that this group can be identified with the group of upper triangular unipotent matrices of order 3, via $(c, b, a) \mapsto \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$. We have the representation of $H$ on $\mathrm{Fun}_{\mathbb{C}}(\mathbb{F})$, given by

$$((c, b, a)f)(x) = \psi(c)\psi(bx)f(x+a).$$

Notice that elements of the form $(c, 0, 0)$ lie in the center of $H$ (and in fact the center is equal to the subgroup of consisting of these elements).

The **Stone-von Neumann** theorem says that there exists a unique, up to isomorphism, irreducible representation of $H$, on which the center $\{(c, 0, 0)\}$ acts by scalars $\psi(c)$ (and this representation is of dimension $|\mathbb{F}|$). To study representations of $H$, we notice that $H$ has commutative subgroups $\{(c, b, 0)\}$ and $\{(0, 0, a\}$, the first of which is normal in $H$, and this makes $H$ a semidirect product of these two subgroups. Therefore we can use the technique of above (let us therefore leave the proof of the Stone-von-Neumann as an exercise, for now).

Therefore, in particular, our representation of $H$ on $\mathrm{Fun}_{\mathbb{C}}(\mathbb{F})$ must be an irreducible representation.

To make the roles of $a$ and $b$ more symmetrical (but one can probably formulate a better explanation), assuming that $|\mathbb{F}|$ is odd, let us slightly re-parametrize $H$:

$$[c, b, a] := (c + \frac{1}{2}ba, b, a).$$

In the new coordinates, the multiplication becomes:

$$[c_2, b_2, a_2][c_1, b_1, a_1] = [c_2 + c_1 + \frac{1}{2}\det\begin{pmatrix} b_1 & a_1 \\ b_2 & a_2 \end{pmatrix}, b_2 + b_1, c_2 + c_1].$$

We now notice that we have an action of $\mathrm{SL}_2(\mathbb{F})$ on $H$ by group automorphisms:

$$A[c, b, a] := [c, (b, a)A^t].$$

Now, consider $A \in \mathrm{SL}_2(\mathbb{F})$ and consider the $H$-representation ${}^A\mathrm{Fun}_{\mathbb{C}}(\mathbb{F})$. Recall that it is $\mathrm{Fun}_{\mathbb{C}}(\mathbb{F})$ as a vector space, but with the $H$-action twisted by $A$. Since the action of $A$ on $H$ leaves the center $\{[c, 0, 0]\}$ of $H$ untouched, ${}^A\mathrm{Fun}_{\mathbb{C}}(\mathbb{F})$ will be again an irreducible representation on which the center acts via $\psi(c)$. By the Stone-von Neumann theorem, ${}^A\mathrm{Fun}_{\mathbb{C}}(\mathbb{F})$ must be isomorphic to $\mathrm{Fun}_{\mathbb{C}}(\mathbb{F})$ as $H$-representations. An isomorphism between them is an invertible operator $T_A : \mathrm{Fun}_{\mathbb{C}}(\mathbb{F}) \to \mathrm{Fun}_{\mathbb{C}}(\mathbb{F})$. It is well-defined up to a scalar, by Schur's lemma. It is a simple exercise that in this way we obtain a **projective representation** of $\mathrm{SL}_2(\mathbb{F})$ on $\mathrm{Fun}_{\mathbb{C}}(\mathbb{F})$. This means that $T_A \circ T_B$ is a scalar multiple of $T_{A \circ B}$, and $T_1$ is a scalar multiple of the identity.

The thus-obtained projective representation of $\mathrm{SL}_2(\mathbb{F})$ on $\mathrm{Fun}_{\mathbb{C}}(\mathbb{F})$ can be thought of as a family of symmetries of the line in which the Fourier transform sits. Namely, one can see that the operator corresponding to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the Fourier transform (up to scalar, as we work up to scalar). One can discuss attempts at making this an actual representation, not just projective one, but we will not do it here.

## 6.7 The group $\mathrm{GL}_2(\mathbb{F})$

Let us fix a finite field $\mathbb{F}$ of odd order, with $q$ elements, where $q$ is an odd positive prime power. We consider the group $G := \mathrm{GL}_2(\mathbb{F})$ of invertible 2 by 2 square matrices over $\mathbb{F}$ (with multiplication in the group being multiplication of matrices). We will work over $\mathbb{C}$. We would like to understand how to construct irreducible representations of $G$, or at least their characters.

### 6.7.1

The number of elements in $G$ is

$$(q^2 - 1)(q^2 - q).$$

### 6.7.2

Let us first figure out the conjugacy classes in $G$ - this is linear algebra. We fix a quadratic extension $\mathbb{E}$ of $\mathbb{F}$, and an element $\epsilon \in \mathbb{E}$ such that $\epsilon \notin \mathbb{F}$ but $\epsilon^2 \in \mathbb{F}$.

1. For the characteristic polynomial $p(x) = (x - c)^2$ for $c \in \mathbb{F}^\times$, matrices with that characteristic polynomial fall into one of the following two conjugacy classes:

(a) Conjugate to $D_{c,c} := \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$.

(b) Conjugate to $J_c := \begin{pmatrix} c & 1 \\ 0 & c \end{pmatrix}$.

2. For the characteristic polynomial $p(x) = (x-c_1)(x-c_2)$ which decomposes into two different linear factors in $\mathbb{F}[x]$ (so $c_1, c_2 \in \mathbb{F}^\times$ with $c_1 \neq c_2$), all matrices with that characteristic polynomial are conjugate (a representative is $D_{c_1,c_2} := \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$).

3. For a characteristic polynomial $p(x)$ which is irreducible in $\mathbb{F}[x]$, with roots $a + b\epsilon, a - b\epsilon \in \mathbb{E}^\times$ (so parametrized by pairs $(a,b) \in \mathbb{F}^2$ for which $b \neq 0$), all matrices with that characteristic polynomial are conjugate to $C_{a+b\epsilon} := \begin{pmatrix} a & b \\ b\epsilon^2 & a \end{pmatrix}$.

In particular, we see that there are

$$\frac{(q-1)(q-2)}{2} + (q-1) + (q-1) + q\frac{q-1}{2} = q^2 - 1$$

conjugacy classes in $G$.

### 6.7.3

Let us denote by $B \subset G$ the subgroup of upper triangular matrices. Let us denote by $U \subset B$ the subgroup of upper triangular unipotent matrices, i.e. those all of whose eigenvalues are 1, i.e. those with 1's on the diagonal. Let us denote by $T \subset B$ the subgroup of diagonal matrices. We have $B = T \ltimes U$, and via this we identify $T \cong B/U$, and so obtain the projection $r : B \to T$.

Notice that $G$ acts naturally on $\mathbb{F}^2$ (by multiplying vectors by matrices), and so acts also on the set of lines in $\mathbb{F}^2$ (by line we mean a one-dimensional sub-vector space). By linear algebra, the action on lines is transitive, and the stabilizer of the line spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is $B$. Therefore we can identify $G/B$ with the set of lines in $\mathbb{F}^2$. In particular, $[G : B] = q + 1$.

Given $\chi \in \mathrm{Ch}(T)$, let us denote by $\widetilde{\chi} \in \mathrm{Ch}(B)$ the composition $B \xrightarrow{r} T \xrightarrow{\chi} \mathbb{C}^\times$. We define the **principal series representation** of $G$:

$$\mathcal{P}_\chi := \mathrm{Ind}_G^B(\mathbb{C}_{\widetilde{\chi}}).$$

Since $[G : B] = q + 1$, this is a representation of dimension $q + 1$.

We use Mackey's theory to study the reducibility and coincidence of the representations $\mathcal{P}_\chi$. For this we need to first figure out representatives for $B\backslash G/B$. Notice that if a matrix $g \in G$ is not in $B$, a Gauss elimination step allows to find

$u \in U$ such that $ug$ has $(1,1)$-component 0. Then, denoting $w := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we have $wug \in B$. Overall, we see that $G$ is the union of the $B$-double cosets $B$ and $BwB$.

Now, we have:

$$\mathrm{Hom}_G(\mathcal{P}_{\chi_1}, \mathcal{P}_{\chi_2}) = \mathrm{Hom}_G(\mathrm{Ind}_G^B(\mathbb{C}_{\widetilde{\chi}_1}), \mathrm{Ind}_G^B(\mathbb{C}_{\widetilde{\chi}_2})) \cong \mathrm{Hom}_B(\mathrm{res}_B^G \mathrm{Ind}_G^B \mathbb{C}_{\widetilde{\chi}_1}, \mathbb{C}_{\widetilde{\chi}_2}) \cong$$
$$\cong \mathrm{Hom}_B(\mathbb{C}_{\widetilde{\chi}_1}, \mathbb{C}_{\widetilde{\chi}_2}) \oplus \mathrm{Hom}_B(\mathrm{Ind}_B^T(\mathbb{C}_{{}^w\chi_1}), \mathbb{C}_{\widetilde{\chi}_2}) \cong$$
$$\cong \mathrm{Hom}_T(\mathbb{C}_{\chi_1}, \mathbb{C}_{\chi_2}) \oplus \mathrm{Hom}_T(\mathbb{C}_{{}^w\chi_1}, \mathbb{C}_{\chi_2})$$

(in the last isomorphism, at the second summand, we used the isomorphism $\mathrm{Ind}_B^T \cong \mathrm{ind}_B^T$ and adjunction). In particular, we see that the dimension of $\mathrm{Hom}_G(\mathcal{P}_{\chi_1}, \mathcal{P}_{\chi_2})$ is the number of elements in the list $(\chi_2, {}^w\chi_2)$ equal to $\chi_1$. In particular, we can conclude that $\mathcal{P}_\chi$ is irreducible if and only if $\chi \neq {}^w\chi$. If $\chi = {}^w\chi$, then $\mathcal{P}_\chi$ is the direct sum of two non-isomorphic irreducibles. There are no repetitions obtained in this way, except for the irreducible $\mathcal{P}_\chi$'s being isomorphic to the $\mathcal{P}_{{}^w\chi}$'s.

Let $\chi$ be such that $\chi = {}^w\chi$. Writing $\chi(\begin{pmatrix} t & 0 \\ 0 & s \end{pmatrix}) = \chi_{(1)}(t) \cdot \chi_{(2)}(s)$ where $\chi_{(1)}, \chi_{(2)} \in \mathrm{Ch}_{\mathbb{C}}(\mathbb{F}^\times)$, the condition simply means $\chi_{(1)} = \chi_{(2)}$. So $\widetilde{\chi}(\begin{pmatrix} t & x \\ 0 & s \end{pmatrix}) = \chi_{(1)}(ts) = \chi_{(1)}(\det(\begin{pmatrix} t & x \\ 0 & s \end{pmatrix}))$. Therefore, the $G$-representation $\mathbb{C}_{\chi_{(1)} \circ \det}$ is a direct summand of $\mathcal{P}_\chi$. Indeed,

$$\mathrm{Hom}_G(\mathbb{C}_{\chi_{(1)} \circ \det}, \mathcal{P}_\chi) \cong \mathrm{Hom}_B(\mathbb{C}_{\chi_{(1)} \circ \det}, \mathbb{C}_{\widetilde{\chi}}),$$

and as the latter is 1-dimensional so is the former, and therefore there is a non-zero $G$-morphism $\mathbb{C}_{\chi_{(1)} \circ \det} \to \mathcal{P}_\chi$ which therefore is an injection. Thus, $\mathcal{P}_\chi$ decomposes into a one-dimensional representation and an irreducible $q$-dimensional representation.

### 6.7.4

Let us calculate the characters of the $\mathcal{P}_\chi$, by using the formula for the character of induction. Here we will think of $G/B$ as the set of lines in the plane, as above. Then the formula for the character says we need to look at lines which our matrix stabilizes, i.e. eigenlines, and the contribution of that line will be $\chi(\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix})$, where $\lambda_1$ is the scalar by which the matrix acts on the line, while $\lambda_2$ is the scalar by which the matrix acts on the quotient of the plane by this line (we give the reader to figure this out!). Therefore, one calculates:

| | $D_{c,c}$ | $J_c$ | $D_{c_1,c_2}$ | $C_{a+b\epsilon}$ |
|---|---|---|---|---|
| $\mathrm{ch}_{\mathcal{P}_\chi}$ | $(q+1) \cdot \chi(D_{c,c})$ | $\chi(D_{c,c})$ | $\chi(D_{c_1,c_2}) + \chi(D_{c_2,c_1})$ | 0 |

We can also calculate, when $\chi = {}^w\chi$, the characters of the 1-dimensional and $q$-dimensional irreducible representations into which $\mathcal{P}_\chi$ decomposes (the notation $\chi_{(1)}$ is as above) - the 1-dimensional we explicitly wrote above, and the character of the $q$-dimensional is the subtraction of the character of the 1-dimensional from that of $\mathcal{P}_\chi$:

| | $D_{c,c}$ | $J_c$ | $D_{c_1,c_2}$ | $C_{a+b\epsilon}$ |
|---|---|---|---|---|
| $\mathrm{ch}_{\mathcal{P}^1_\chi}$ | $\chi_{(1)}(c^2)$ | $\chi_{(1)}(c^2)$ | $\chi_{(1)}(c_1c_2)$ | $\chi_{(1)}(a^2-b^2\epsilon^2)$ |
| $\mathrm{ch}_{\mathcal{P}^q_\chi}$ | $q\cdot\chi_{(1)}(c^2)$ | $0$ | $\chi_{(1)}(c_1c_2)$ | $-\chi_{(1)}(a^2-b^2\epsilon^2)$ |

### 6.7.5

We therefore obtained $(q-1)\cdot 2+\frac{(q-1)(q-2)}{2}$ irreducible representations, which we will call those **of the principal series**, so there are still $\frac{q(q-1)}{2}$ missing ones. The sum of squares of dimensions of the missing irreducible representations is seen to be $\frac{q(q-1)}{2}\cdot(q-1)^2$, so it is tempting to think that all the missing irreducible representations have dimension $q-1$. We will now show this.

Let $V$ be a finite-dimensional $G$-representation. We consider the $U$-isotypic components of $V$ - given $\psi\in\mathrm{Ch}_{\mathbb{C}}(U)$ we have the isotypic component

$$V_{U,\psi} := \{v\in V \mid uv=\psi(u)v \mid \forall u\in U\}.$$

Notice that $T$ normalizes $U$, so acts on $U$ by conjugation and therefore also on $\mathrm{Ch}_{\mathbb{C}}(U)$. Given $t\in T$, we have $tV_{U,\psi}=V_{U,{}^t\psi}$ and in particular $\dim V_{U,\psi}=\dim V_{U,{}^t\psi}$. Notice that $T$ acts transitively on $\mathrm{Ch}_{\mathbb{C}}(U)\smallsetminus\{1\}$ and therefore, fixing some $\psi_0\in\mathrm{Ch}_{\mathbb{C}}(U)\smallsetminus\{1\}$, we see that

$$\dim V=(q-1)\dim V_{U,\psi_0}+\dim V_{U,1}.$$

Now, we claim that given an irreducible $G$-representation, $E$ is of the principal series if and only if $E_{U,1}\neq 0$. Indeed, $E$ is of the principal series if and only if $\mathrm{Hom}_G(E,\mathrm{Ind}_G^B(\mathbb{C}_{\widetilde{\chi}}))\neq 0$ for some $\chi\in\mathrm{Ch}_{\mathbb{C}}(T)$. We have

$$\mathrm{Hom}_G(E,\mathrm{Ind}_G^B(\mathbb{C}_{\widetilde{\chi}}))\cong\mathrm{Hom}_B(E,\mathbb{C}_{\widetilde{\chi}})\cong\mathrm{Hom}_B(E_{U,1},\mathbb{C}_{\widetilde{\chi}})=\mathrm{Hom}_T(E_{U,1},\mathbb{C}_\chi)$$

(the second identification is because $\mathbb{C}_{\widetilde{\chi}}=(\mathbb{C}_{\widetilde{\chi}})_{U,1}$ and the third identification is because $U$ acts trivially on both representaitons, so we can consider them as representations of $B/U\cong T$). Now, one finish by noticing that given a $T$-representation $W$, we have $W=0$ if and only if $\mathrm{Hom}_T(W,\mathbb{C}_\chi)=0$ for all $\chi$. So, we can proceed now to conclude that for an irreducible $G$-representation $E$ which is not of the principal series, we have $\dim E=(q-1)\dim E_{U,\psi_0}$. In particular, we have $\dim E\geq q-1$, and so from the above numerics there is no choice but to conclude that all those irreducible representations have dimension exactly $q-1$.

### 6.7.6

There are various ways to construct the irreducible representations which are not of the principal series (they are called **cuspidal**), but here we would like just to construct their characters, rather than the actual representations.

The basic idea is that we have used characters of $T$ to define the principal series. To define the cuspidal representations, one would like to use the characters of a different subgroup

$$K := \{\begin{pmatrix} a & \epsilon^2 b \\ b & a \end{pmatrix} : \ (a,b) \in \mathbb{F}^2 \smallsetminus \{(0,0)\}\}.$$

Note that $K$ is similar to $T$,[6] in being the subgroup stabilizing the $\mathbb{E}$-lines spanned by $\begin{pmatrix} \epsilon \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -\epsilon \\ 1 \end{pmatrix}$ (the latter is in fact redundant, by considering conjugation), considering the natural action of $G$ on $\mathbb{E}^2$. Note also that if we identify $\mathbb{F}^2$ with $\mathbb{E}$ by $(a,b) \mapsto a + \epsilon b$, then each element of $K$ acts on $\mathbb{E}$ by multiplication by a scalar in $\mathbb{E}$ ($\begin{pmatrix} a & \epsilon^2 b \\ b & a \end{pmatrix} \in K$ acts as $a + \epsilon b \in \mathbb{E}^{\times}$), and in this way $K$ is identified with $\mathbb{E}^{\times}$.

Let us emphasize that in our previous construction,would we take $\mathrm{Ind}_G^T \mathbb{C}_\chi$, we would have obtained representations that are too big to analyse, but $\mathrm{Ind}_G^B(\mathbb{C}_{\widetilde{\chi}})$ were already good (almost always irreducible). For $K$, one doesn't have a subgroup analogous to $B$ to perform this reduction of dimension.

So we will now do the following strange thing: we will hope that the difference $\mathrm{ch}_{\mathrm{Ind}_G^T(\mathbb{C}_\chi)} - \mathrm{ch}_{\mathrm{Ind}_T^B(\mathbb{C}_{\widetilde{\chi}})}$ is also somehow the correct difference, between the too big and the desired, in the case of $K$. Let us compute this difference. To compute the character of $\mathrm{Ind}_G^T(\mathbb{C}_\chi)$ we interpret $G/T$ as (isomorphic to) the set of ordered pairs of non-equal lines in $\mathbb{F}^2$.

| | $D_{c,c}$ | $J_c$ | $D_{c_1,c_2}$ | $C_{a+b\epsilon}$ |
|---|---|---|---|---|
| $\mathrm{ch}_{\mathrm{Ind}_G^T(\mathbb{C}_\chi)}$ | $(q+1)q \cdot \chi(D_{c,c})$ | $0$ | $\chi(D_{c_1,c_2}) + \chi(D_{c_2,c_1})$ | $0$ |
| $\mathrm{ch}_{\mathcal{P}_\chi}$ | $(q+1) \cdot \chi(D_{c,c})$ | $\chi(D_{c,c})$ | $\chi(D_{c_1,c_2}) + \chi(D_{c_2,c_1})$ | $0$ |
| $\mathrm{ch}_{\mathrm{Ind}_G^T(\mathbb{C}_\chi)} - \mathrm{ch}_{\mathcal{P}_\chi}$ | $(q+1)(q-1) \cdot \chi(D_{c,c})$ | $-\chi(D_{c,c})$ | $0$ | $0$ |

We already see a good sign - we see that the difference $\mathrm{ch}_{\mathrm{Ind}_G^T(\mathbb{C}_\chi)} - \mathrm{ch}_{\mathrm{Ind}_T^B(\mathbb{C}_{\widetilde{\chi}})}$ only depends on the values of $\chi$ on $Z \subset G$, the subgroup of scalar matrices (the center of $G$) - Let us therefore write, given $\theta \in \mathrm{Ch}_{\mathbb{C}}(Z)$, $d_\theta \in \mathrm{Fun}_{\mathbb{C}}(G)^{cl}$ for the function in the last row above, where we replace $\chi(D_{c,c})$ by $\theta(D_{c,c})$. This is good because $K$ also contains $Z$, and so every character of $K$ gives by restriction a character of $Z$. So we can now hope that, given a character $\chi \in \mathrm{Ch}_{\mathbb{C}}(K)$,

$$\mathrm{ch}_{\mathrm{Ind}_G^K(\mathbb{C}_\chi)} - d_{\chi|_Z}$$

[6] In the theory of algebraic groups, one would say that $T$ is (the $\mathbb{F}$-points of) a split torus, while $K$ is (the $\mathbb{F}$-points of) a non-split torus, which splits over $\mathbb{E}$.

is the character of an irreducible representation.

Let us now calculate the above expression. It is easy to see that the action of $G$ on $\mathbb{E}$-lines in $\mathbb{E}^2$ which do not contain non-zero vectors in $\mathbb{F}^2$ (i.e. $\mathbb{E}$-liens "not defined over $\mathbb{F}$") is transitive and, as mentioned above, $K$ is the stabilizer of the $\mathbb{E}$-line spanned by $\begin{pmatrix} \epsilon \\ 1 \end{pmatrix}$. Therefore we can think of $G/K$ as the $G$-set of $\mathbb{E}$-lines not defined over $\mathbb{F}$.

| | $D_{c,c}$ | $J_c$ | $D_{c_1,c_2}$ | $C_{a+b\epsilon}$ |
|---|---|---|---|---|
| $\mathrm{ch}_{\mathrm{Ind}_G^K(\mathbb{C}_\chi)}$ | $(q^2-q)\cdot\chi(D_{c,c})$ | $0$ | $0$ | $\chi(C_{a+b\epsilon})+\chi(C_{a-b\epsilon})$ |
| $\mathrm{ch}_{\mathrm{Ind}_G^K(\mathbb{C}_\chi)} - d_{\chi\|Z}$ | $-(q-1)\cdot\chi(D_{c,c})$ | $\chi(D_{c,c})$ | $0$ | $\chi(C_{a+b\epsilon})+\chi(C_{a-b\epsilon})$ |

We forgot to tell before the following. Denote by $R(G)$ the $\mathbb{Z}$-submodule of $\mathrm{Fun}_{\mathbb{C}}(G)^{cl}$ spanned by characters of irreducible representations. By linear independence of characters, in fact the characters of irreducible representations form a $\mathbb{Z}$-basis for $R(G)$. Now we have the following useful observation: Given $f \in R(G)$, we have $\langle f, f\rangle = 1$ if and only if $f$ is either the character of an irreducible representation or minus the character of an irreducible representation. Indeed, letting $E_1, \ldots, E_r$ be an exhaustive list of the irreducible representations of $G$, write $f = \sum_i n_i \cdot \mathrm{ch}_{E_i}$ (where $n_i \in \mathbb{Z}$). Then $\langle f, f\rangle = \sum_i n_i^2$. This is equal to 1 if and only if all $n_i$ except one of them are equal to 0, and that one is equal to 1 or $-1$. Clearly one can distinguish between the character of an irreducible representation and minus the character of an irreducible representation by looking at the value at 1 - it is equal to the dimension for the former, and to minus the dimension for the latter.

We therefore set $\nu_\chi := -(\mathrm{ch}_{\mathrm{Ind}_G^K(\mathbb{C}_\chi)} - d_{\chi|Z})$ and hope that it is the character of an irreducible representation. For that, we need to check that $\langle \nu_\chi, \nu_\chi\rangle = 1$. Let us denote $w_K := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Notice that ${}^{w_K}\chi(a+b\epsilon) = \chi(a-b\epsilon)$. We compute:

$$\langle \nu_\chi, \nu_\chi\rangle = \frac{1}{(q^2-1)(q^2-q)}\Bigg((q-1)\cdot(q-1)^2 + (q-1)\cdot(q+1)\cdot(q-1)\cdot 1^2 +$$

$$+\frac{1}{2}\sum_{a+b\epsilon\in\mathbb{E}^\times\smallsetminus\mathbb{F}^\times}(q^2-q)\cdot\Big(\frac{\chi}{{}^{w_K}\chi}(a+b\epsilon) + \frac{{}^{w_K}\chi}{\chi}(a+b\epsilon)+2\Big)\Bigg) =$$

$$= \frac{1}{(q-1)^2q(q+1)}\Bigg((q-1)^2\cdot 2q + \frac{1}{2}(q-1)q\sum_{a+b\epsilon\in\mathbb{E}^\times}\Big(\frac{\chi}{{}^{w_K}\chi}(a+b\epsilon)+\frac{{}^{w_K}\chi}{\chi}(a+b\epsilon)+2\Big) - \frac{1}{2}(q-1)q\cdot 4(q-1)\Bigg) =$$

$$= 1 + \frac{1}{q^2-1}\cdot\frac{1}{2}\sum_{a+b\epsilon\in\mathbb{E}^\times}\Big(\frac{\chi}{{}^{w_K}\chi}(a+b\epsilon) + \frac{{}^{w_K}\chi}{\chi}(a+b\epsilon)\Big)$$

which is equal to 1 if $\chi$ is such that ${}^{w_K}\chi \neq \chi$ and to 2 otherwise. It is an exercise to show that the determinant map $\mathbb{E}^\times \to \mathbb{F}^\times$ is surjective, and its kernel consists of elements of the form $\frac{a+b\epsilon}{a-b\epsilon}$, so that ${}^{w_K}\chi = \chi$ if and only if $\chi$ factors via the determinant map. Therefore there are $q(q-1)$ characters $\chi$ for which ${}^{w_K}\chi \neq \chi$. For such $\chi$, $\nu_\chi$ is the character of an irreducible representation of dimension $q-1$. One sees that $\nu_\chi = \nu_{\chi'}$ if and only if $\chi' = \chi$ or $\chi' = {}^{w_K}\chi$ (either from looking at the character values, using linear independence of characters of $K$, or by calculating $\langle \nu_\chi, \nu_{\chi'} \rangle$). Therefore we obtain the characters of $\frac{q(q-1)}{2}$ different irreducible representation of dimension $q-1$, which therefore must be all the remaining irreducible representations.

Let us summarize the character table of $G = \mathrm{GL}_2(\mathbb{F})$:

| | $D_{c,c}$ | $J_c$ | $D_{c_1,c_2}$ | $C_{a+b\epsilon}$ |
|---|---|---|---|---|
| $\mathbb{C}_{\theta\circ\det}$ $(\theta \in \mathrm{Ch}_{\mathbb{C}}(\mathbb{F}^\times))$ | $\theta(c^2)$ | $\theta(c^2)$ | $\theta(c_1c_2)$ | $\theta(a^2-b^2\epsilon^2)$ |
| $\mathrm{ch}_{\mathcal{P}_\chi}$ $(\chi \in \mathrm{Ch}_{\mathbb{C}}(T), {}^{w}\chi \neq \chi)$ | $(q+1)\cdot\chi(D_{c,c})$ | $\chi(D_{c,c})$ | $\chi(D_{c_1,c_2})+\chi(D_{c_2,c_1})$ | 0 |
| $\mathrm{ch}_{\mathcal{P}^q_\chi}$ $(\chi \in \mathrm{Ch}_{\mathbb{C}}(T), {}^{w}\chi = \chi)$ | $q\cdot\chi(D_{c,c})$ | 0 | $\chi(D_{c,c})$ | $-\chi(C_{a+b\epsilon}C_{a-b\epsilon})$ |
| $\nu_\chi$ $(\chi \in \mathrm{Ch}_{\mathbb{C}}(K), {}^{w_K}\chi \neq \chi)$ | $(q-1)\cdot\chi(D_{c,c})$ | $-\chi(D_{c,c})$ | 0 | $-(\chi(C_{a+b\epsilon})+\chi(C_{a-b\epsilon}))$ |

## 6.8 Brauer's induction theorem

Throughout this subsection, we fix a finite group $G$, and work over $\mathbb{C}$.

### 6.8.1

For a subring $A \subset \mathbb{C}$, we denote by $R_A(G)$ the $A$-span of the characters of finite-dimensional $G$-representations, in $\mathrm{Fun}_{\mathbb{C}}(G)^{cl}$. This is an $A$-algebra, with pointwise addition and multiplication. Moreover, the characters of irreducible $G$-representations form an $A$-basis for $R_A(G)$. Given a subgroup $H \subset G$, we have $A$-linear maps

$$R_A(H) \underset{\mathrm{res}^G_H}{\overset{\mathrm{Ind}^H_G}{\rightleftarrows}} R_A(G)$$

(the map $\mathrm{Ind}^H_G$ is the restriction to $R_A(H)$ of the map that we called ${}^{\mathrm{fun}}Ind^H_G$, we for notational simplicity abbreviate the notation here).

In particular, we have $R_{\mathbb{C}}(G) = \mathrm{Fun}_{\mathbb{C}}(G)^{cl}$. In the other extreme, we set $R(G) := R_{\mathbb{Z}}(G)$ (this is sometimes called the **representation ring** of $G$). Elements of $R(G)$ are sometimes called **virtual characters**. As we have already mentioned above, a virtual character $\chi \in R(G)$ satisfies $\langle \chi, \chi \rangle = 1$ if and only if it is the character of an irreducible representation, or minus the character of an irreducible representation.

### 6.8.2

**Artin's induction theorem** states that if we are given a family of subgroups $(H_i)_{i\in I}$ of $G$ such that every element in $G$ is conjugate to an element in some $H_i$, then the map

$$\oplus_{i\in I}\mathrm{Ind}_G^{H_i} : \bigoplus_{i\in I} R_{\mathbb{Q}}(H_i) \to R_{\mathbb{Q}}(G)$$

is surjective. Indeed, it is clear by linear algebra that this map is surjective if and only if the analogous map

$$\oplus_{i\in I}\mathrm{Ind}_G^{H_i} : \bigoplus_{i\in I} R_{\mathbb{C}}(H_i) \to R_{\mathbb{C}}(G)$$

is surjective. Using our non-degenerate symmetric bilinear form, to check that this map is surjective it is enough to check that if $f \in R_{\mathbb{C}}(G)$ is orthogonal to the image of the map, then $f = 0$. Indeed, if $f$ is orthogonal to the image then $\langle f, \mathrm{Ind}_G^{H_i} f'\rangle = 0$ for all $i \in I$ and $f' \in R_{\mathbb{C}}(H_i)$. By Frobenius reciprocity we therefore have $\langle \mathrm{res}^G_{H_i} f, f'\rangle = 0$ for all $i \in I$ and $f' \in R_{\mathbb{C}}(H_i)$. By the non-degeneracy of our forms, we obtain $\mathrm{res}^G_{H_i} f = 0$ for all $i \in I$. In other words, our function $f$ vanishes on all the subgroups $H_i$. Since $f$ is a class function, by the assumption on the $H_i$'s it follows that $f = 0$.

One can take $(H_i)_{i\in I}$ to simply consist of the cyclic subgroups in $G$. Each $R_{\mathbb{Q}}(H_i)$ is spanned by characters in $\mathrm{Ch}_{\mathbb{C}}(H_i)$, and therefore we obtain:

**Theorem 6.8** (Main version of Artin's induction theorem)**.** *Let $V$ be a finite-dimensional representation of $G$. Then there exist cyclic subgroups $H_1, \dots, H_r \subset G$ and characters $\chi_i \in \mathrm{Ch}_{\mathbb{C}}(H_i)$ such that $\mathrm{ch}_V$ is a rational linear combination of $\mathrm{Ind}_G^{H_1}(\chi_1), \dots \mathrm{Ind}_G^{H_r}(\chi_r)$.*

Brauer's induction theorem states:

**Theorem 6.9** (Brauer's induction theorem)**.** *Let $V$ be a finite-dimensional representation of $G$. Then there exist subgroups $H_1, \dots, H_r \subset G$ and characters $\chi_i \in \mathrm{Ch}_{\mathbb{C}}(H_i)$ such that $\mathrm{ch}_V$ is a $\mathbb{Z}$-linear combination of $\mathrm{Ind}_G^{H_1}(\chi_1), \dots \mathrm{Ind}_G^{H_r}(\chi_r)$.*

The proof of Brauer's induction theorem is more complicated. The main application is to show that Artin $L$-functions are products of Hecke $L$-functions and their inverses, therefore deducing that Artin $L$-functions have a meromorphic continuation to the whole complex plane.

### 6.8.3

Recall the inductive definition of being a nilpotent group. A group $H$ is called 0-nilpotent if it is the trivial group. It is called $r$-nilpotent, for $r \geq 1$, if $H/Z(H)$ is $(r-1)$-nilpotent. The group $H$ is called **nilpotent** if it is $r$-nilpotent for some $r \in \mathbb{Z}_{\geq 0}$. It is easy to see that subgroups and quotient groups of nilpotent groups are nilpotent. A product of two nilpotent groups is also nilpotent. Recall that any $p$-group is nilpotent.

**Lemma 6.10.** *Let $H$ be a nilpotent group. If $H$ is not abelian, then $H$ contains a normal abelian subgroup which is not central.*

*Proof.* Since $H$ is not abelian, we have $Z(H) \neq H$. Consider an element $h \in H$ which maps to a non-trivial element of $Z(H/Z(H))$ under the projection $H \to H/Z(H)$. Then Denote by $K$ the subgroup of $H$ generated by $Z(H)$ and $h$. Then $K$ is not central, it is abelian, and it is also normal in $H$ (since it is the preimage under $H \to H/Z(H)$ of a central subgroup in $H/Z(H)$). □

**Lemma 6.11.** *Let $H$ be a finite nilpotent group. Let $E$ be an irreducible representation of $H$. Then there exists a subgroup $K \subset H$ and a character $\chi \in \mathrm{Ch}_{\mathbb{C}}(K)$ such that $E \cong \mathrm{Ind}_H^K(\mathbb{C}_\chi)$.*

*Proof.* We proceed by induction on $|H|$.

We first reduce to the case when the representation $E$ of $H$ is faithful. Let $L$ be the kernel of the representation. If $L \neq 1$, then considering $E$ as a $H/L$-representation, by induction, we can find a subgroup $\widetilde{K} \subset H/L$ and a character $\chi \in \mathrm{Ch}_{\mathbb{C}}(\widetilde{K})$ such that $E$ is isomorphic to $\mathrm{Ind}_{H/L}^{\widetilde{K}}(\mathbb{C}_{\widetilde{\chi}})$ as $H/L$-representations. We leave to the reader to see that in fact then $E$ is isomorphic to $\mathrm{Ind}_H^K(\mathbb{C}_\chi)$ as $H$-representations, where $K$ is the preimage in $H$ of $\widetilde{K}$ under $H \to H/L$, and $\chi$ is the composition of $K \to \widetilde{K}$ with $\widetilde{\chi}$.

So we now assume that $E$ is a faithful $H$-representation. We can also assume that $H$ is not abelian, since if $H$ is abelian the claim is clear. By the above lemma, there exists a non-central abelian normal subgroup $L \subset H$. Let us recall that previously we saw that either the $E$ is $L$-isotypical, or there exists $L \subset K \subset G$ such that $K \neq G$ and $E \cong \mathrm{Ind}_G^K F$ for some irreducible $K$-representation $F$. In the latter case, using induction we can find a subgroup $K' \subset K$ and a character $\chi \in \mathrm{Ch}_{\mathbb{C}}(K')$ such that $F \cong \mathrm{Ind}_K^{K'}(\mathbb{C}_\chi)$. Then $E \cong \mathrm{Ind}_H^{K'}(\mathbb{C}_\chi)$ and we are done. It is left to see that the former case is not possible. Indeed, if $E$ is $L$-isotypical then, since $L$ is abelian, we have that $L$ acts on $E$ by scalars. Then the image of $L$ in $\mathrm{GL}(E)$ is in the center of the image of $H$ in $\mathrm{GL}(E)$. Since the representation is faithful, we obtain that $L$ is in the center of $H$, contradicting $L$ being non-central in $H$. □

### 6.8.4

Given a prime $p$, let us say that a finite group $H$ is ***p*-elementary** if it is isomorphic to a product of a $p$-group with a cyclic group. We say that $H$ is **elementary** if it is $p$-elementary for some prime $p$. Clearly, an elementary group is nilpotent.

**Theorem 6.12** (Brauer's induction theorem)**.** *The map*

$$\oplus_{\substack{H \subset G \\ \text{elementary}}} \mathrm{Ind}_G^H : \bigoplus_{\substack{H \subset G \\ \text{elementary}}} R(H) \to R(G)$$

*is surjective.*

Notice that Theorem 6.12, coupled with Lemma 6.11, gives Theorem 6.9. The remainder of the subsection will be devoted to proving Theorem 6.12.

### 6.8.5

We will have to use the following **projection formula**:

**Lemma 6.13.** *Let $H \subset G$ be a subgroup. Let $V$ be a $G$-representation and $W$ a $H$-representation. Then*

$$\mathrm{ind}_G^H (W \otimes \mathrm{res}_H^G V) \cong \mathrm{ind}_G^H W \otimes V.$$

*Proof.* Let us leave this as an exercise for now. □

Let $I \subset R(G)$ be the image of our map (we want to show that $I = R(G)$.

**Reduction 6.14.** *It is enough to see that $1 \in I$.*

*Proof.* By the above projection formula, $I$ is an ideal in $R(G)$. Therefore $1 \in I$ will imply $I = R(G)$. □

### 6.8.6

For subrings $A, B \subset \mathbb{C}$, let us denote by $R_A^B(G) \subset R_A(G)$ the subset consisting of functions all of whose values lie in $B$. Clearly $R_A^B(G)$ is a subring of $R_A(G)$, and the restriction and induction operators preserve $R_A^B(-)$. Let us denote by $I_A^B$ the image of the map analogous to the previous one:

$$\oplus_{\substack{H \subset G \\ \text{elementary}}} \mathrm{Ind}_G^H : \bigoplus_{\substack{H \subset G \\ \text{elementary}}} R_A^B(H) \to R_A^B(G).$$

We have $I = I_{\mathbb{Z}}^{\mathbb{C}}$.

Let us set $A$ to be the subring of $\mathbb{C}$ generated by all $|G|$-th roots of unity.

**Reduction 6.15.** *It is enough to see that $1 \in I_A^{\mathbb{C}}$.*

*Proof.* Since $A$ is finitely generated as a $\mathbb{Z}$-module, all elements of $A$ are integral. Therefore $A \cap \mathbb{Q} = \mathbb{Z}$. Therefore $A/\mathbb{Z}$ is a torsion free finitely generated $\mathbb{Z}$-module. Therefore we can write $A = \mathbb{Z} \oplus A'$ for some $\mathbb{Z}$-submodule $A' \subset A$. We consider the corresponding projection $p : A \to \mathbb{Z}$, and consider the projection $P : R_A(G) \to R_{\mathbb{Z}}(G)$ given by writing an element as an $A$-linear combination of irreducible characters and applying our projection to the coefficients. Now, if $1 \in I_A^{\mathbb{C}}$ then we can write $1 = \sum_i a_i \cdot \mathrm{ch}_{Ind_G^{H_i} E_i}$ where $a_i \in A$ and $H_i$ are elementary subgroups in $G$. Then

$$1 = P(1) = P(\sum_i a_i \cdot \mathrm{ch}_{\mathrm{Ind}_G^{H_i} E_i}) = \sum_i P(a_i \cdot \mathrm{ch}_{\mathrm{Ind}_G^{H_i} E_i}) = \sum_i p(a_i) \cdot \mathrm{ch}_{\mathrm{Ind}_G^{H_i} E_i} \in I.$$

The justification of the last equality is as follows. We can write $\mathrm{ch}_{\mathrm{Ind}_G^{H_i} E_i} = \sum_j n_j \cdot \mathrm{ch}_{F_j}$ where $n_j \in \mathbb{Z}$ and $F_j$ are distinct irreducible representations of $G$. Then $a_i \cdot \mathrm{ch}_{\mathrm{Ind}_G^{H_i} E_i} = \sum_j (a_i n_j) \cdot \mathrm{ch}_{F_j}$ and therefore by definition

$$P(a_i \cdot \mathrm{ch}_{\mathrm{Ind}_G^{H_i} E_i}) = \sum_j p(a_i n_j) \cdot \mathrm{ch}_{F_j} = p(a_i) \sum_j \cdot n_j \cdot \mathrm{ch}_{F_j} = p(a_i) \cdot \mathrm{ch}_{\mathrm{Ind}_G^{H_i} E_i}.$$

□

We also make the following obvious reduction:

**Reduction 6.16.** *It is enough to see that* $1 \in I_A^{\mathbb{Z}}$.

*Proof.* This is clear in view of the previous reduction, as $I_A^{\mathbb{Z}} \subset I_A^{\mathbb{C}}$. □

### 6.8.7

We now consider $R_A^{\mathbb{Z}}(G) \subset R_{\mathbb{C}}^{\mathbb{Z}}(G)$. Notice that $R_{\mathbb{C}}^{\mathbb{Z}}(G)$ is a finitely generated free abelian group (it is simply $\mathrm{Fun}_{\mathbb{Z}}(G)^{cl}$). We will use the following lemma:

**Lemma 6.17.** *Let $L$ be a finitely generated free abelian group, let $M \subset L$ be a subgroup and let $\ell \in L$. If $\ell \in M + p^k L$ for all primes $p$ and $k \in \mathbb{Z}_{\geq 1}$, then $\ell \in M$.*

We now have the following reduction:

**Reduction 6.18.** *It is enough to see that for every prime $p$ and $k \in \mathbb{Z}_{\geq 1}$, there exists $f_{p,k} \in I_A^{\mathbb{Z}}$ such that all the values of $1 - f_{p,k}$ are divisible by $p^k$.*

*Proof.* We apply the lemma to $L := R_{\mathbb{C}}^{\mathbb{Z}}(G)$, $M := I_A^{\mathbb{Z}}$ and $\ell := 1$. □

### 6.8.8

We reduce to the following:

**Reduction 6.19.** *It is enough to see that for every prime $p$ there exists $f_p \in I_A^{\mathbb{Z}}$ all of whose values are prime to $p$.*

*Proof.* Given in addition $k \in \mathbb{Z}_{\geq 1}$, by Euler's theorem all the values of $1 - f_p^{\phi(p^k)}$ are divisible by $p^k$, so $f_{p,k} := f_p^{\phi(p^k)} \in I_A^{\mathbb{Z}}$ is as desired in the previous reduction. □

### 6.8.9

Let us call an element $h \in H$ in a finite group **$p$-regular** if the order of $h$ is prime to $p$, and **$p$-torsion** if the order of $h$ is a power of $p$.

**Lemma 6.20.** *Let $H$ be a finite group. Then any $h \in H$ can be written uniquely $h = h_{p\text{-r}} h_{p\text{-t}}$ where $h_{p\text{-r}}$ is $p$-regular, $h_{p\text{-t}}$ is $p$-torsion, and $h_{p\text{-r}}$ and $h_{p\text{-t}}$ commute.*

We now reduce the claim further:

**Reduction 6.21.** *It is enough to see the following. Let $p$ be a prime and let $g \in G$ be a $p$-regular element. Then there exists $f_g \in I_A^{\mathbb{Z}}$ such that $f_g(x)$ is prime to $p$ if if $x_{p\text{-r}}$ is conjugate to $g$ and $f_g(x) = 0$ otherwise.*

*Proof.* If we sum the functions $f_g$ as in the statement, running over representatives $g$ of conjugacy classes of $p$-regular elements in $G$, we clearly obtain a function $f_p \in I'$ all of whose values are prime to $p$, as desired in the previous reduction. □

### 6.8.10

Let us consider a $p$-Sylow subgroup $S$ of $Z_G(g)$. Then clearly $E := S \cdot \langle g \rangle$ is a $p$-elementary subgroup of $G$. Denoting by $o(g)$ the order of $g$, let us consider $f := o(g) \cdot \delta_{\{g\}} \in R_A^{\mathbb{Z}}(\langle g \rangle)$. We leave to the reader to figure out that indeed $F$ lies in $R_A(\langle g \rangle)$ (recall the Fourier theory expression of $f$ as a linear combination of characters). We now denote by $f_1 \in R_A^{\mathbb{Z}}(E)$ the pullback of $f$ under the projection $E \to \langle g \rangle$. So $f_1 = o(g) \cdot \delta_{Sg}$. Finally, let us denote $f_g := \mathrm{Ind}_G^E(f_1) \in I_A^{\mathbb{Z}} \subset R_A^{\mathbb{Z}}(G)$. We claim that $f_g$ is as desired, i.e. $f_g(x) = 0$ if $x_{p\text{-r}}$ is conjugate to $g$ and $f_g(x)$ is prime to $p$ otherwise. Let us calculate:

$$f_g(x) = o(g) \cdot |\{y \in G/E \mid y^{-1}xy \in Sg\}| = |\{y \in G/S \mid y^{-1}xy \in Sg\}|.$$

If $x_{p\text{-r}}$ is not conjugate to $g$, then clearly the expression is equal to 0. Suppose that $x_{p\text{-r}}$ is conjugate to $g$. By conjugating we can assume without loss of generality that $x_{p\text{-r}} = g$ and so $x \in Z_G(g)$. Further conjugating inside $Z_G(g)$, by Sylow's theorem, we can assume without loss of generality that $x \in E$. So write $x = sg$ with $s \in S$. Then $y \in G$ satisfying $y^{-1}xy \in Sg$ must satisfy $y^{-1}gy = g$ i.e. $y \in Z_G(g)$. The remaining condition is $y^{-1}sy \in S$. Hence we want to show that

$$|\{y \in Z_G(g)/S \mid y^{-1}sy \in S\}|$$

is prime to $p$. We can interpret this as

$$|\{yS \in Z_G(g)/S \mid s(yS) = yS\}|.$$

In other words, we want to show that the number of fixed points of the action of $s$ on $G_Z(g)/S$ is prime to $p$. Since $s$ is $p$-torsion and $|Z_G(g)/S|$ is prime to $p$, this follows from the following lemma:

**Lemma 6.22.** *Let $X$ be a finite set such that $|X|$ is prime to $p$. Let $\sigma : X \to X$ be a self-bijection whose order is a $p$-power. Then the number of fixed points of $\sigma$ on $X$ is prime to $p$.*

We are done!

# 7 Some additional topics

## 7.1 Gelfand pairs, Gelfand-Tseitlin basis

Throughout, $G$ is a finite group and $k$ is an algebraically closed field of characteristic zero.

### 7.1.1

Let $K \subset G$ be a subgroup.

**Definition 7.1.** We say that $(G, K)$ is a **Gelfand pair** if for every irreducible $G$-representation $E$ we have $[res_K^G E : k] \leq 1$ (where here $k$ is the trivial representation of $K$). We say that $(G, K)$ is a **strong Gelfand pair** if for every irreducible $G$-representation $E$ and every irreducible $K$-representation $F$ we have $[res_K^G E : F] \leq 1$.

**Remark 7.2.** Notice that $[res_K^G E : k] = \dim_k E^K$.

Our motivating example is:

**Claim 7.3.** *$(S_n, S_{n-1})$ is a strong Gelfand pair.*

We will prove this in the end of the discussion. Let us say that a **axis system** for a $m$-dimensional vector space $V$ is an unordered $m$-tuple $\{\ell_1, \ldots, \ell_m\}$ of one-dimensional subspaces of $V$ such that $V = \ell_1 \oplus \ldots \oplus \ell_m$. A corollary of $(S_n, S_{n-1})$ being a strong Gelfand pair is that every irreducible $S_n$-representation has a canonical axis system (the "Gelfand-Tseitlin basis"). Indeed, starting with an irreducible $S_n$-representation $E$, we decompose it as a direct sum of irreducible $S_{n-1}$-representations. Each isomorphism class of irreducible $S_{n-1}$-representations appears in the decomposition at most once, since $(S_n, S_{n-1})$ is a strong Gelfand pair. Therefore, in fact, the decomposition is canonical (because the $S_{n-1}$-irreducible summands are the same as the $S_{n-1}$-isotypic components). Now we continue, and decompose each $S_{n-1}$-irreducible summand into a direct sum of irreducible $S_{n-2}$-representations. And so on, we continue in this fashion, in the end obtaining a decomposition into irreducible $S_1$-representaitons, which are simply one-dimensional subspaces.

**Example 7.4.** *Let us consider the standard irreducible $S_4$-representation, on*

$$E := \{v = (x_i)_{1 \leq i \leq 4} \mid \sum x_i = 0\}.$$

### 7.1.2

Let $K \subset G$ be a subgroup.

**Definition 7.5.** Let us denote by $\mathcal{H}(G, K) \subset k[G]$ the subset consisting of $K$-biinvariant elements, i.e. elements $d$ satisfying $\delta_k d = d$ and $d \delta_k = d$ for all $k \in K$.

**Definition 7.6.** Let us define $e_K := \frac{1}{|K|} \sum_{k \in K} \delta_k \in k[G]$.

**Exercise 7.1.** *We have $e_K^2 = e_K$ and $e_K \in \mathcal{H}(G,K)$. Also, $\mathcal{H}(G,K)$ is a non-unital subalgebra of $k[G]$ (i.e. it is a $k$-vector subspace of $k[G]$ closed under multiplication), and it is itself a unital algebra with unit $e_K$. Finally, we have $e_K d e_K \in \mathcal{H}(G,K)$ for all $d \in k[G]$.*

**Definition 7.7.** The algebra $\mathcal{H}(G,K)$ is called the **Hekce algebra** (of $(G,K)$).

### 7.1.3

Let $V$ be a $G$-representation. Then $V^K$ is a sub-$\mathcal{H}(G,K)$-module of $V$. Indeed, let $d \in \mathcal{H}(G,K)$ and $v \in V^K$. Then for any $k \in K$ we have

$$kdv = (\delta_k d)v = dv$$

and therefore $dv \in V^K$. Notice also that the unit $e_K$ of $\mathcal{H}(G,K)$ acts on $V^K$ by identity, so that $V^K$ is indeed a unital $\mathcal{H}(G,K)$-module.

**Definition 7.8.** Let $E$ be an irreducible $G$-representation. We say that $E$ is **$K$-spherical** if $E^K \neq 0$.

**Claim 7.9.** *If $E$ is a $K$-spherical $G$-representation then $E^K$ is an irreducible $\mathcal{H}(G,K)$-module.*

*Proof.* Let $E$ be a $K$-spherical irreducible $G$-representation. Let $N \subset E^K$ be a sub-$\mathcal{H}(G,K)$-module. We claim that $(k[G] \cdot N) \cap E^K = N$. Indeed, the inclusion from right to left is evident. Let $v \in (k[G] \cdot N) \cap E^K$. So we can write $v = \sum d_i n_i$ with $d_i \in k[G]$ and $n_i \in N$. Then

$$v = e_K v = \sum e_K d_i n_i = \sum e_K d_i e_K n_i \in N$$

(the last inclusion is since $e_K d_i e_K \in \mathcal{H}(G,K)$). Therefore, if $N \subset E^K$ is non-zero, we have $k[G] \cdot N = E$ since $E$ is $G$-irreducible and therefore

$$N = (k[G] \cdot N) \cap E^K = E \cap E^K = E^K,$$

showing that $E^K$ is an irreducible $\mathcal{H}(G,K)$-module. □

**Corollary 7.10.** *Suppose that $\mathcal{H}(G,K)$ is commutative. Then $(G,K)$ is a Gelfand pair.*

*Proof.* If $\mathcal{H}(G,K)$ is commutative, then all irreducible $\mathcal{H}(G,K)$-modules are one-dimensional. Hence, given an irreducible $G$-representation $E$, if $E$ is not $K$-spherical then $E^K = 0$ and if $E$ is $K$-spherical then by the previous Claim $E^K$ is an irreducible $\mathcal{H}(G,K)$-module and hence $\dim_k(E^K) = 1$. □

**Claim 7.11** (Gelfand trick)**.** *Suppose that we have an anti-involution $r : G \to G$ (i.e. $r(g_1 g_2) = r(g_2) r(g_1)$ and $r(r(g)) = g$) which preserves all $K$-double cosets in $G$. Then $\mathcal{H}(G,K)$ is commutative.*

*Proof.* Define $R : k[G] \to k[G]$ by $R(\sum c_g \delta_g) := \sum c_g \delta_{r(g)}$. Then $R$ is an anti-involution of $k[G]$. Notice that if $d \in \mathcal{H}(G,K)$ then $R(d) = d$, because writing $d = \sum c_g \delta_g$ we have $R(d) = \sum c_g \delta_{r(g)} = \sum c_{r(g)} \delta_g$ but since $r(g)$ lies in the same $K$-double coset as $g$, we have $c_{r(g)} = c_g$ and therefore $R(d) = d$. Thus, $R$ restricts to an anti-involution of $\mathcal{H}(G,K)$, which is the identity map. This shows that $\mathcal{H}(G,K)$ is commutative: $d_1 d_2 = R(d_1 d_2) = R(d_2)R(d_1) = d_2 d_1$. □

**Example 7.12.** *Let us consider $(G,K) = (S_n, S_{n-1})$. We consider the anti-involution of $S_n$ given by $g \mapsto g^{-1}$. It is not hard to see, thinking about cycle decomposition, that this anti-involution preserves, in fact, $S_{n-1}$-conjugacy classes - for any $h \in S_n$ there exists $k \in S_{n-1}$ such that $h^{-1} = khk^{-1}$. In particular, this anti-involution preserves $K$-double cosets and hence $\mathcal{H}(S_n, S_{n-1})$ is commutative and hence $(S_n, S_{n-1})$ is a Gelfand pair.*

### 7.1.4

**Claim 7.13.** *$(G,K)$ is a strong Gelfand pair if and only if $(G \times K, \Delta K)$ is a Gelfand pair, where $\Delta K = \{(k,k) : k \in K\} \subset G \times K$.*

*Proof.* The irreducible $G \times K$-representations are of the form $F \otimes E^*$ where $F$ is an irreducible $G$-representation and $E$ is an irreducible $K$-representation. Then

$$[\mathrm{res}_{\Delta K}^{G\times K}(F \otimes E^*) : k] = \dim_k (F \otimes E^*)^K = \dim_k \mathrm{Hom}_K(E,F) = [res_K^G F : E]$$

and from this the claim follows. □

Let us denote by $k[G]_K$ (probably should find a better name) the subalgebra of $k[G]$ consisting of $d$ for which $\delta_k d \delta_k^{-1} = d$ for all $k \in K$.

**Lemma 7.14.** *The algebras $\mathcal{H}(G \times K, \Delta K)$ and $k[G]_K$ are isomorphic.*

*Proof.* Let us define a map

$$\phi : \mathcal{H}(G \times K, \Delta K) \to k[G]_K$$

by sending

$$\sum_{(g,k) \in G \times K} c_{(g,k)} \cdot \delta_{(g,k)} \mapsto \sum_{g \in G} c_{(g,1)} \cdot \delta_g.$$

We check that we indeed land in $k[G]_K$:

$$\delta_k \left( \sum_{g \in G} c_{(g,1)} \cdot \delta_g \right) \delta_{k^{-1}} = \sum_{g \in G} c_{(g,1)} \cdot \delta_{kgk^{-1}} = \sum_{g \in G} c_{(k^{-1}gk,1)} \cdot \delta_g = \sum_{g \in G} c_{(g,1)} \cdot \delta_g.$$

Now we check whether $\phi$ is an algebra morphism (**it is not!**):

$$\phi \left( \left( \sum_{(g,k) \in G \times K} c_{(g,k)} \cdot \delta_{(g,k)} \right) \cdot \left( \sum_{(g,k) \in G \times K} c'_{(g,k)} \cdot \delta_{(g,k)} \right) \right) = \phi \left( \sum_{\substack{g_1, g_2 \in G \\ k_1, k_2 \in K}} c_{(g_1,k_1)} c'_{(g_2,k_2)} \delta_{(g_1 g_2, k_1 k_2)} \right) =$$

$$= \sum_{g_1,g_2\in G}\sum_{k\in K} c_{(g_1,k)}c'_{(g_2,k^{-1})}\delta_{g_1g_2} = \sum_{g_1,g_2\in G}\sum_{k\in K} c_{(g_1k^{-1},1)}c'_{(kg_2,1)}\delta_{g_1g_2} =$$

$$= \sum_{k\in K}\sum_{g_1,g_2\in G} c_{(g_1k^{-1},1)}c'_{(kg_2,1)}\delta_{(g_1k^{-1})(kg_2)} = \sum_{k\in K}\sum_{g_1,g_2\in G} c_{(g_1,1)}c'_{(g_2,1)}\delta_{g_1g_2} =$$

$$= |K|\cdot\left(\sum_{g\in G} c_{(g,1)}\delta_g\right)\cdot\left(\sum_{g\in G} c'_{(g,1)}\delta_g\right) = |K|\cdot\phi\left(\sum_{(g,k)\in G\times K} c_{(g,k)}\cdot\delta_{(g,k)}\right)\cdot\phi\left(\sum_{(g,k)\in G\times K} c'_{(g,k)}\cdot\delta_{(g,k)}\right)$$

Now we construct an inverse to $\phi$:

$$\psi : k[G]_K \to \mathcal{H}(G\times K, \Delta K)$$

given by sending

$$\sum_{g\in G} c_g\cdot\delta_g \mapsto \sum_{(g,k)\in G\times K} c_{k^{-1}g}\delta_{(g,k)}.$$

One readily checks that $\psi\circ\phi = \mathrm{id}$ and $\phi\circ\psi = \mathrm{id}$.

Therefore, we see that $|K|\cdot\phi$ is an algebra isomorphism as desired! □

**Corollary 7.15.** *Suppose that $k[G]_K$ is commutative. Then $(G,K)$ is a strong Gelfand pair.*

*Proof.* If $k[G]_K$ is commutative then by the above lemma we have that $\mathcal{H}(G\times K,\Delta K)$ is commutative, and therefore by a previous result $(G\times K,\Delta K)$ is a Gelfand pair, and hence by a previous result $(G,K)$ is a strong Gelfand pair. □

**Claim 7.16** (Gelfand trick)**.** *Suppose that we have an anti-involution $r : G\to G$ which preserves all $K$-conjugacy classes in $G$. Then $k[G]_K$ is commutative (and hence $(G,K)$ is a strong Gelfand pair).*

*Proof.* The proof is as before: $r$ induces an anti-involution $R$ of $k[G]$, which restricts to the identity on $k[G]_K$, and therefore $k[G]_K$ is commutative. □

**Corollary 7.17.** *$(S_n,S_{n-1})$ is a Strong Gelfand pair.*

*Proof.* We consider the anti-involution $g\mapsto g^{-1}$ of $S_n$. As we have already mentioned, it preserves $S_{n-1}$-conjugacy classes, and so the condition of the Gelfand trick is fulfilled. □

# 8 Representation theory of compact groups (3l qSh hmzlg)

## 8.1 Topological groups, Haar measures

### 8.1.1

For us, a compact space will always mean a Hausdorff compact space. A locally compact space will mean a Hausdorff locally compact space. We will assume all topological spaces are Hausdorff, unless stated otherwise.

**Definition 8.1.** A **topological group** is a set $G$ equipped both with a group structure and a topology, such that the multiplication map $G \times G \to G$ and the inverse map $G \to G$ are continuous.

**Example 8.2.** *Any group can be given the discrete topology, making it a locally compact topological group.*

**Example 8.3.** *The group $\mathrm{GL}_n(\mathbb{R})$ of invertible $(n \times n)$-matrices over $\mathbb{R}$, equipped with the usual topology (subspace topology with respect to the inclusion $\mathrm{GL}_n(\mathbb{R}) \subset M_n(\mathbb{R})$, where $M_n(\mathbb{R})$ is given the usual topology on a finite-dimensional real vector space), is a locally compact topological group.*

**Example 8.4.** *Given a topological group $G$ and a closed subgroup $H \subset G$ (here "closed" refers to the topology - being a closed subset), $H$ itself becomes naturally a topological group. We can therefore obtain a lot of examples from the previous example - $\mathrm{SL}_n(\mathbb{R})$ (matrices with determinant $1$), $O_n$ (orthogonal matrices), and so on. Notice that $O_n$ is a compact group.*

**Example 8.5.** *We similarly have the locally compact topological group $\mathrm{GL}_n(\mathbb{C})$ and its compact subgroup $U_n$ consisting of unitary matrices.*

**Example 8.6.** *Given a topological group $G$ and a closed normal subgroup $H \subset G$, the quotient group $G/H$ equipped with the quotient topology is a topological group. A very important example is $\mathbb{R}/\mathbb{Z}$ (where $\mathbb{R}$ is a group with respect to addition). This group has many other models (it is the **circle group**). For example, it is isomorphic to $SO_2$, the group of $(2 \times 2)$-matrices which are orthogonal with determinant $1$ (the group of **rotations of the plane**) - the isomorphism $\mathbb{R}/\mathbb{Z} \to SO_2$ is given by sending $r + \mathbb{Z} \mapsto \begin{pmatrix} cos(r) & -sin(r) \\ sin(r) & cos(r) \end{pmatrix}$. Also, it is isomorphic to the subgroup $\mathbb{C}^{\times,1} \subset \mathbb{C}^{\times}$ consisting of the complex numbers with length $1$ - the isomorphism $\mathbb{R}/\mathbb{Z} \to \mathbb{C}^{\times,1}$ is given by sending $r + \mathbb{Z} \mapsto e^{2\pi i r}$).*

**Example 8.7.** *One also has the locally compact topological groups such as $\mathrm{GL}_n(\mathbb{Q}_p)$ (where $\mathbb{Q}_p$ is the field of $p$-adic numbers) and $\mathrm{GL}_n(\mathbb{A})$ (where $\mathbb{A}$ is the ring of adeles) which play important roles in number theory.*

**Example 8.8.** *There are also natural examples of topological groups which are not locally compact. For example, one can take a locally compact topological group $G$, a compact topological space $X$, and consider the group of continuous maps from $X$ to $G$ (with pointwise multiplication), with the open-compact topology.*

### 8.1.2

Let $X$ be a locally compact topological space. One has two dual approaches to what a measure on $X$ is. In one approach, a measure associates to nice enough subsets of $X$ values in $\mathbb{R}$ (the measure of a subset). In the other approach, a measure associates to continuous functions with compact support on $X$ values in $\mathbb{R}$ (the integral of a function). Let us take the second approach.

**Definition 8.9.** We denote by $C(X)$ the complex vector space of complex-valued continuous functions on $X$. We denote by $C_c(X) \subset C(X)$ the subspace consisting of functions with compact support.

**Definition 8.10.** A **signed Radon measure** on $X$ is a functional $\mu : C_c(X) \to \mathbb{C}$ satisfying the following: Let $K \subset X$ be a compact subset and let $(f_n)_{n\in\mathbb{Z}_{\geq 1}}$ be a sequence of functions in $C_c(X)$, such that the support of each $f_n$ lies in $K$, and converging uniformly to a function $f \in C_c(X)$. Then $(\mu(f_n))_{n\in\mathbb{Z}_{\geq 1}}$ converges to $\mu(f)$. The signed Radon measure $\mu$ is said to be a **Radon measure** if $\mu(f) \in \mathbb{R}_{\geq 0}$ for any $f \in C_c(X)$ such that $f(x) \in \mathbb{R}_{\geq 0}$ for all $x \in X$. We will denote by $\mathcal{M}(X)$ the vector space of signed Radon measures on $X$.

**Example 8.11.** *Associating to $f \in C_c(\mathbb{R})$ the Riemann integral $\int_{-\infty}^{\infty} f(x)dx$, we obtain a Radon measure on $\mathbb{R}$. Fixing $f_0 \in C(\mathbb{R})$, we can also consider the signed Radon measure $f \mapsto \int_{-\infty}^{\infty} f(x)f_0(x)dx$.*

**Example 8.12.** *Let $x \in X$. We can consider the Radon measure on $X$, sending $f$ to $f(x)$. This is called **Dirac's delta** $\delta_x$.*

One has a natural structure of a $C(X)$-module on $\mathcal{M}(X)$ (where $C(X)$ is an algebra with respect to pointwise multiplication). Namely, given $f \in C(X)$ and $\mu \in \mathcal{M}(X)$, one defines $f\mu \in \mathcal{M}(X)$ by $(f\mu)(f') := \mu(ff')$.

### 8.1.3

Let $\theta$ be an automorphism of a locally compact topological space $X$ (i.e. a homeomorphism of $X$ to itself) Then we obtain an automorphism of the vector space $C_c(X)$, given by sending $f$ to $(x \mapsto f(\theta^{-1}(x))$, and then an automorphism of the vector space $\mathcal{M}(X)$, given by sending $\mu$ to $(f \mapsto \mu(\theta^{-1}(f))$.

In particular, if $G$ is a locally compact topological group, we can consider the action of $G$ on itself by left and right translations - the left action given by $L_{g_0}g := g_0 g$ and the right action given by $R_{g_0}g := gg_0^{-1}$. Each such $L_{g_0}$ and $R_{g_0}$ is an automorphism of the topological space $G$, and therefore by the above procedure we obtain corresponding actions of $G$ on $C_c(G)$ and on $\mathcal{M}(G)$. So, for example, $(R_{g_0}\mu)(f) = \mu((g \mapsto f(gg_0^{-1})))$.

### 8.1.4

Let $G$ be a locally compact topological group.

**Definition 8.13.** A **(left) Haar measure** on $G$ is a Radon measure $\int$ on $G$, which is non-zero and left $G$-invariant - meaning that $L_{g_0}\int = \int$ for all $g_0 \in G$. Similarly, a **right Haar measure** on $G$ is a non-zero right $G$-invariant Radon measure on $G$.

**Theorem 8.14** (Haar theorem). *There exists a Haar measure on $G$, and any two Haar measures on $G$ differ by a real positive scalar. In fact, any left $G$-invariant signed Radon measure on $G$ differs by a complex scalar from a Haar* *measure. Also, a Haar measure $\int$ on $G$ is nowhere-vanishing, in the sense that if $f \in C_c(G)$ satisfies $f(g) \in \mathbb{R}_{\geq 0}$ for all $g \in G$ and $f \neq 0$, then $\int f \in \mathbb{R}_{>0}$.*

*Proof.* We omit the proof. □

**Example 8.15.** *On $\mathbb{R}$, a group with respect to addition, the usual Riemann integral Radon measure considered above (sending $f$ to $\int_{-\infty}^{\infty} f(x)dx$) is a Haar measure.*

**Example 8.16.** *Let $G$ be a group with the discrete topology. Then a Haar measure is given by $f \mapsto \sum_{g \in G} f(g)$ (note that the sum is well-defined since only finitely many summands are non-zero). This is called the* ***counting measure****. So, we have a normalization of the Haar measure in this case, by requiring the integral of a function equal to 1 at some point and to 0 at the rest of the points to be 1.*

**Example 8.17.** *Let us consider the circle group $\mathbb{R}/\mathbb{Z}$. A Haar measure on it can be for example described as sending $f$ to $\int_0^1 f(x+\mathbb{Z})dx$.*

**Example 8.18.** *If $G$ is compact, we can normalize the Haar measure by requiring the integral of the constant function 1 to be equal to 1. Notice that if $G$ is a finite group, it is a compact group and also a discrete group, and we obtain two different normalizations of the Haar measure.*

### 8.1.5

Of course, all the previous can also be said for right Haar measures. Notice that if $\int$ is a left Haar measure, then $f \mapsto \int(g \mapsto f(g^{-1}))$ is a right Haar measure, and vice versa. To discuss the possible difference between left and right Haar measures, we proceed as follows.

Let $\theta$ be an automorphism of the topological group $G$ (so, $\theta$ is a group automorphism and also a homeomorphism). Let $\int$ be a left Haar measure on $G$. Then one easily checks that $\theta \int$ is also a left Haar measure on $G$. By the uniqueness of Haar measure, there exists a scalar $\Delta(\theta) \in \mathbb{R}^{\times}_{>0}$ (called the **modulus of** $\theta$) such that $\theta \int = \Delta(\theta) \cdot \int$. One checks easily that $\Delta(\mathrm{id}) = 1$ and $\Delta(\theta_1\theta_2) = \Delta(\theta_1)\Delta(\theta_2)$, in other words that $\Delta : \mathrm{Aut}(G) \to \mathbb{R}^{\times}_{>0}$ is a group homomorphism.

In particular, given $g \in G$ let $\theta_g$ denote the automorphism of $G$ given by $x \mapsto gxg^{-1}$. We abbreviate $\Delta(g) := \Delta(\theta_g)$. We obtain therefore a group homomorphism $\Delta : G \to \mathbb{R}^{\times}_{>0}$, commonly referred to as the **modulus function**.

**Lemma 8.19.** *The modulus function $\Delta : G \to \mathbb{R}^{\times}_{>0}$ is continuous.*

*Proof.* We omit the proof. □

Now, let $\int$ be a left Haar measure on $G$. We calculate:

$$(R_{g_0} \int)(f) = \int (g \mapsto f(gg_0^{-1})) = \int (g \mapsto f(g_0 g_0^{-1} g g_0^{-1}) = \int (g \mapsto (\theta_{g_0^{-1}} f)(g_0^{-1} g)) =$$

$$= \int \theta_{g_0^{-1}} f = (\theta_{g_0} \int)(f) = \Delta(g_0) \int f,$$

i.e. we obtain $R_{g_0} \int = \Delta(g_0) \int$. We therefore deduce, in particular, that a left Haar measure is also a right Haar measure if and only if $\Delta = 1$. In that case one says that $G$ is **unimodular**.

**Exercise 8.1.** *Given a left Haar measure* $\int$ *on* $G$*, show that* $f \mapsto \int(g \mapsto f(g)\Delta(g)^{-1})$ *is a right Haar measure on* $G$*.*

**Claim 8.20.** *If* $G$ *is compact, or discrete, or abelian, then* $G$ *is unimodular.*

*Proof.* It is an easy exercise that $\mathbb{R}^{\times}_{>0}$ does not admit compact closed subgroups except the trivial subgroup $\{1\}$. If $G$ is compact, the image of $\Delta$ is a compact subgroup of $\mathbb{R}^{\times}_{>0}$, hence trivial, i.e. $\Delta = 1$. Next, if $G$ is discrete, one sees directly that the counting measure is both left and right invariant. Finally, if $G$ is abelian, then it is also clear from definitions that being left or right invariant are the same thing. □

**Exercise 8.2.** *Let us consider the locally compact topological group* $G := \mathbb{R} \rtimes \mathbb{R}^{\times}$ *(where the semidirect product is formed by letting the multiplicative group act on the additive group by multiplication). Show that* $G$ *is not unimodular. Calculate the modulus function.*

## 8.2 Finite-dimensional representations

### 8.2.1

Recall that a **norm** $||-||$ on a $\mathbb{C}$-vector space $V$ is a function $||-|| : V \to \mathbb{R}_{\geq 0}$ such that $||cv|| = |c| \cdot ||v||$, $||v_1 + v_2|| \leq ||v_1|| + ||v_2||$, and $||v|| = 0$ if and only if $v = 0$. A norm on $V$ induces a metric on $V$, given by $d(v_1, v_2) := ||v_2 - v_1||$. A metric, in its turn, induces a topology on $V$. By a **normed space** we mean a $\mathbb{C}$-vector space equipped with a norm. We have the following basic claim:

**Claim 8.21.** *Let* $V$ *be a finite-dimensional* $\mathbb{C}$*-vector space. Then there exist norms on* $V$*, and any two norms on* $V$ *induce the same topology.*

Let us give another example of a normed vector space. Let $X$ be a compact topological space. We define a norm on $C(X)$, the **supremum norm**, by setting $||f||_{sup} := sup_{x \in X} |f(x)|$.

### 8.2.2

**Definition 8.22.** Let $G$ be a locally compact topological group. Let $V$ be a normed space. A **(continuous) representation** of $G$ on $V$ is a representation in our old sense $G \times V \to V$ such that, additionally, the map $G \times V \to V$ is continuous. If we want to emphasize, we will call a representation for which $G \times V \to V$ is not necessarily continuous an **abstract representation**.

**Definition 8.23.** Let $G$ be a locally compact topological group and $V$ a finite-dimensional vector space. A **(continuous) representaiton** of $G$ on $V$ is a continuous representation of $G$ on $V$ when $V$ is considered as normed with any norm (this will not depend on the norm, as the topology we obtain from a norm on $V$ does not depend on the norm chosen).

**Exercise 8.3.** *Show that an abstract representation of $G$ on a finite-dimensional vector space $V$ is continuous if and only if for every $v \in V$ the map $G \to V$ given by $g \mapsto gv$ is continuous. Also, if and only if the associated group homomorphism $G \to \mathrm{GL}(V)$ is continuous.*

All the basic definitions (subrepresentations, quotient representations, irreducible representations, direct sums of finite families of representations, tensor product of representations, dual representations and so on) can be made here. Schur's lemma holds - given irreducible $E, F$, we have $\mathrm{Hom}_G(E, F) = 0$ if $E$ is not isomorphic to $F$ and $\dim \mathrm{Hom}_G(E, F) = 1$ if $E$ is isomorphic to $F$ (explicitly, $\mathrm{Hom}_G(E, E) = \mathbb{C} \cdot \mathrm{Id}_E$). But, of course, we don't have semisimplicity in this generality, as we already didn't have it for infinite discrete groups, which are in particular locally compact topological groups. However, we will see now that if $G$ is compact, semisimplicity holds as it did for finite groups.

**Remark 8.24.** Let $X$ be a locally compact topological space and $\mu \in \mathcal{M}(X)$. Then it is a simple "formal" exercise to see that, given a finite-dimensional $\mathbb{C}$-vector space $V$, and denoting by $C_c(G; V)$ the space of continuous functions $G \to V$ with compact support, there exists a unique linear map $C_c(G; V) \to V$, which we also denote by $\mu$ by slight abuse of notation, such that given $f \in C_c(G)$ and $v \in V$, and denoting by $f \cdot v \in C_c(G; V)$ the function sending $g$ to $f(g) \cdot v$, we have $\mu(f \cdot v) = \mu(f) \cdot v$. Moreover, given a linear map $T : V_1 \to V_2$ of finite-dimensional $\mathbb{C}$-vector spaces, we will have $\mu(T \circ f) = T(\mu(f))$ for all $f \in C_c(G; V_1)$.

**Claim 8.25.** *Assume that $G$ is compact. Then any finite-dimensional $G$-representation $V$ is semisimple, i.e. given a $G$-subrepresentation $W \subset V$, there exists a $G$-subrepresentation $U \subset V$ such that $V = W \oplus U$.*

*Proof.* Let $\int$ denote the Haar measure on $G$ normalized by $\int 1 = 1$. Let $P_0 : V \to V$ be a projection onto $W$. We define a new linear map $P : V \to V$ as follows:

$$P(v) := \int \left( g \mapsto gP_0(g^{-1}v) \right).$$

One immediately sees that if $v \in W$ then $P(v) = v$ and that the image of $P$ lies in $W$ - in other words, that $P$ is a projection onto $W$. One also checks that $P$ is a $G$-morphism (here one uses $\int$ being a left Haar measure). Therefore we will have a direct sum decomposition $V = W \oplus \mathrm{Ker}(P)$ where $\mathrm{Ker}(P)$ is a $G$-subrepresentation of $V$. $\square$

### 8.2.3

So, if $G$ is a compact group, the theory of finite-dimensional representations of it is similar to that of a finite group. However, a difference is that one will have infinitely many irreducible representations, up to isomorphism.

**Claim 8.26.** *Let $G$ be an abelian locally compact group. Then every irreducible finite-dimensional representation of $G$ is 1-dimensional.*

*Proof.* Let $V$ be an irreducible finite-dimensional $G$-representation. Let $g \in G$. Let $\lambda$ an eigenvalue of the action of $g$ on $V$, and let $V_{g,\lambda} \subset V$ be the corresponding eigenspace. An easy exercise shows that the action of any element in $G$ preserves $V_{g,\lambda}$ (here one uses the commutativity of $G$, of course), i.e. $V_{g,\lambda}$ is a $G$-subrepresentation of $V$. Since $V$ is irreducible, we must have $V_{g,\lambda} = V$. In other words, we showed that any $g \in G$ acts on $V$ by a scalar. Therefore any subspace of $V$ is a subrepresentation. Since $V$ is irreducible, we must have that $V$ is 1-dimensional. $\square$

**Exercise 8.4.** *So, given an abelian locally compact group $G$ we obtain, as before, a bijection between the set of continuous characters $G \to \mathbb{C}^\times$ and the set of isomorphism classes of irreducible finite-dimensional $G$-representations.*

**Exercise 8.5.** *The continuous characters of $G := \mathbb{R}/\mathbb{Z}$ are given by $\chi_n(x+\mathbb{Z}) := e^{2\pi i(nx)}$, for $n \in \mathbb{Z}$.*

### 8.2.4

Let $G$ be a locally compact topological group. We say that a finite-dimensional $G$-representation $V$ is **unitary**, if it is equipped with an inner product $\langle -,- \rangle$ such that $\langle gv_1, gv_2 \rangle = \langle v_1, v_2 \rangle$ for all $v_1, v_2 \in V$ and $g \in G$ (such an inner product is said to be **$G$-invariant**). We say that $V$ is **unitarizable** if it admits a unitary structure.

**Lemma 8.27.** *Assume that $G$ is compact. Then any finite-dimensional $G$-representation $V$ is unitarizable.*

*Proof.* Denote by $\int$ the normalized Haar measure on $G$. One chooses any inner product $\langle -,- \rangle_0$ on $V$, and then defines a new inner product

$$\langle v_1, v_2 \rangle := \int (g \mapsto \langle gv_1, gv_2 \rangle).$$

Then one checks that $\langle -,- \rangle$ is a $G$-invariant inner product. $\square$

**Exercise 8.6.** *Find a non-unitarizable finite-dimensional representation of a locally compact topological group.*

Given a finite-dimensional $G$-representation $V$, recall that we have the dual representation $V^*$. We also define the conjugate representation $\overline{V}$ as being $V$ with the same addition, the same $G$-action, but the multiplication by scalar is changed to be $c *_{\text{new}} v := \bar{c} *_{\text{old}} v$. An inner product on $V$ can be seen as an isomorphism $\overline{V} \to V^*$ (given by sending $w$ to $(v \mapsto \langle v, w\rangle)$), with the extra property of positivity. If $V$ is a unitary representation of $G$ then the corresponding isomorphism of vector spaces $\overline{V} \to V^*$ is in fact an isomorphism of $G$-representations. As a corollary, we see that for any finite-dimensional representation $V$ of a compact group $G$, the $G$-representations $V^*$ and $\overline{V}$ are isomorphic (not necessarily canonically).

In particular, a finite-dimensional irreducible $G$-representation $E$ admits a unique unitary structure, up to scalar (if it admits one). Indeed, we just explained that unitary structures inject into $G$-isomorphisms $\overline{E} \to E^*$. It is clear that $\overline{E}$ and $E^*$ are irreducible $G$-representations. Hence by Schur's lemma all isomorphisms between them differ by a scalar.

We also have the following:

**Lemma 8.28.** *Let $V$ be a finite-dimensional unitary $G$-representation. Let $E, F \subset V$ be non-isomorphic irreducible sub-representations. Then $E$ and $F$ are orthogonal.*

*Proof.* Let us consider the $G$-morphism $\overline{E} \to F^*$ given by sending $v \in E$ to the functional on $F$ sending $w \in F$ to $\langle w, v\rangle$. Notice that $\overline{E}$ and $F^*$ are non-isomorphic irreducible $G$-representations (because, since $E$ is unitary, $\overline{E} \cong E^*$, and so $\overline{E} \cong F^*$ would imply $E^* \cong F^*$ and so $E \cong F$). Therefore by Schur's lemma our $G$-morphism must be equal to zero. This precisely shows that $E$ and $F$ are orthogonal. □

### 8.2.5

Let $G$ be a compact group, with normalized Haar measure $\int$. We have a theory of characters as before. Namely, for a finite-dimensional representation $V$ of $G$ we define a function $\text{ch}_V \in C_c(G)$ by setting

$$\text{ch}_V(g) := \text{Tr}(g \curvearrowright V).$$

Then $\text{ch}_V \in C_c(G)^{cl}$, where

$$C_c(G)^{cl} := \{f \in C_c(G) \mid f(hgh^{-1}) = f(g) \ \forall g, h \in G\} \subset C_c(G).$$

One checks that $\text{ch}_{V_1 \oplus V_2} = \text{ch}_{V_1} + \text{ch}_{V_2}$. One checks that $\text{ch}_{V_1 \otimes V_2} = \text{ch}_{V_1} \cdot \text{ch}_{V_2}$. Also, $\text{ch}_{\overline{V}} = \overline{\text{ch}_V}$ (where we define $\overline{f}(g) := \overline{f(g)}$). By what we saw above, we therefore also have $\text{ch}_{V^*} = \overline{\text{ch}_V}$. Finally, one has the averaging operator

$$\text{Av}_G^V : V \to V \qquad \text{Av}_G^V(v) := \int (g \mapsto gv),$$

which one checks to be a projection onto the subspace of invariants

$$V^G := \{v \in V \mid gv = v \ \forall g \in G\} \subset V.$$

Using this avergaing operator, one checks as before that $\dim V^G = \int \mathrm{ch}_V$. Finally, we have the inner product on $C_c(G)$ given by

$$\langle f_1, f_2 \rangle := \int f_1 \cdot \overline{f_2}$$

and one has the basic equality

$$\dim \mathrm{Hom}_G(V_1, V_2) = \dim \mathrm{Hom}(V_1, V_2)^G = \int \mathrm{ch}_{\mathrm{Hom}(V_1,V_2)} =$$

$$= \int \mathrm{ch}_{V_1^* \otimes V_2} = \int \overline{\mathrm{ch}_{V_1}} \mathrm{ch}_{V_2} = \langle \mathrm{ch}_{V_2}, \mathrm{ch}_{V_1} \rangle.$$

One deduces the orthogonality relation - for irreducible finite-dimensional $G$-representations $E$ and $F$, one has $\langle \mathrm{ch}_E, \mathrm{ch}_F \rangle = 1$ if $E$ and $F$ are isomorphic and $\langle \mathrm{ch}_E, \mathrm{ch}_F \rangle = 0$ otherwise.

One has the theory of isotypic components as before, and given a finite-dimensional irreducible $G$-representation $E$, the projection $P_E : V \to V$ onto the $E$-isotypic component is given by

$$v \mapsto \dim E \cdot \int (g \mapsto \overline{\mathrm{ch}_E(g)} \cdot gv).$$

### 8.2.6

We now want to find irreducible representations of the group $G := SU(2)$ of unitary $(2 \times 2)$-matrices with determinant 1. We have a natural representation of $G$ on $\mathbb{C}^2$. Let $P_n$, for $n \in \mathbb{Z}_{\geq 0}$, denote the vector space of functions on $\mathbb{C}^2$ spanned by $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto x^i y^j$ with $i, j \in \mathbb{Z}_{\geq 0}$, $i + j = n$. So $\dim P_n = n + 1$. We have a natural action of $G$ on $P_n$ (given by $(gf)(v) := f(g^{-1}v)$). It will turn out that $(P_n)_{n \in \mathbb{Z}_{\geq 0}}$ are exactly all the irreducible $G$-representations, up to isomorphism.

We will now prove it using a formula which we will not currently deduce. Denote by $T \subset G$ the subgroup of diagonal matrices. By linear algebra (diagnolizability of unitary operators), any element of $G$ is conjugate to an element in $T$. Therefore a function $f \in C_c(G)^{cl}$ is determined by its values on $T$. Let us denote by $\alpha$ the following character of $T$:

$$\alpha(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}) = t$$

(where $t \in \mathbb{C}^{\times,1}$). Then $(\alpha^n)_{n \in \mathbb{Z}}$ are precisely all the characters of $T$. The character of any finite-dimensional $T$-representation can be written uniquely as $\sum_{n \in \mathbb{Z}} c_n \cdot \alpha^n$, where $c_n \in \mathbb{Z}_{\geq 0}$ and all the $c_n$'s except finitely many are equal to 0 (here, of course $c_n$ is the dimension of the $\alpha^n$-isotypic component). In particular, given a finite-dimensional $G$-representation $V$, $\mathrm{ch}_V$ is determined by the knowledge of the family $(c_n)_{n\in\mathbb{Z}}$ for which $(\mathrm{ch}_V)|_T = \mathrm{ch}_{\mathrm{res}^G_T V} = \sum_{n\in\mathbb{Z}} c_n \cdot \alpha^n$.

Let us find this decomposition for $\mathrm{ch}_{P_n}$. It is clear that $T$ acts on a basis element $x^i y^j$ of $P_n$ by the character $\alpha^{j-i}$. Therefore we have

$$\mathrm{ch}_{P_n} = \alpha^{-n} + \alpha^{-n+2} + \ldots + \alpha^n.$$

Now, as $f \in C_c(G)^{cl}$ is determined by $f|_T$, one should be able to calculate $\int_G f$ just from the data of $f|_T$. Since we have two compact groups now, $G$ and $T$, let us denote by $\int_G$ and $\int_T$ the respective noramlized Haar measures.

**Theorem 8.29** (Weyl's integration formula)**.** *Let $f \in C_c(G)^{cl}$. We have*

$$\int_G f = \frac{1}{2}\int_T f|_T \cdot |\alpha - \alpha^{-1}|^2.$$

*Proof.* We don't prove the theorem now. □

We can now see that $P_n$ are irreducible (of course, this can also be shown in various other ways). Namely, we calculate:

$$\langle \mathrm{ch}_{P_n}, \mathrm{ch}_{P_n}\rangle = \int_G |\mathrm{ch}_{P_n}|^2 = \frac{1}{2}\int_T |(\alpha^{-n} + \ldots + \alpha^n)(\alpha - \alpha^{-1})|^2 =$$

$$= \frac{1}{2}\int_T |\alpha^{n+1} - \alpha^{-n-1}|^2 = 1.$$

We would like to see now that any irreducible $G$-representation is isomorphic to some $P_n$. Let $V$ be a finite-dimensional $G$-representation. Write $(\mathrm{ch}_V)|_T = \sum_{n\in\mathbb{Z}} c_n \cdot \alpha^n$. We first claim that $c_n = c_{-n}$ for all $n \in \mathbb{Z}$. Indeed, consider the element $w := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in G$. This element normalizes $T$, and sends $\alpha$ to $\alpha^{-1}$. Therefore the action of $w$ on $V$ sends the $(T, \alpha^n)$-isotypic component to the $(T, \alpha^{-n})$-isotypic component. In particular, those have the same dimension. Assume now that $V$ is irreducible. Then $\langle \mathrm{ch}_V, \mathrm{ch}_V\rangle = 1$. Let us see what this condition means:

$$1 = \langle \mathrm{ch}_V, \mathrm{ch}_V\rangle = \int_G |\mathrm{ch}_V|^2 = \frac{1}{2}\int_T |(\sum_{n\in\mathbb{Z}} c_n \cdot \alpha^n)(\alpha - \alpha^{-1})|^2 =$$

$$= \frac{1}{2}\int_T |\sum_{n\in\mathbb{Z}}(c_{n-1} - c_{n+1}) \cdot \alpha^n|^2 = \frac{1}{2}\sum_{n\in\mathbb{Z}} |c_{n-1} - c_{n+1}|^2.$$

It is easy to see from this that the sequence $(c_n)_{n\in\mathbb{Z}}$ must be identical to that coming from some $P_n$.

**8.2.7**

Let us talk also about $G := SO(3)$. Set $\widetilde{G} := SU(2)$. We think of $\widetilde{G}$ embedded in $\mathrm{GL}_2(\mathbb{C})$ in the usual way. Let us consider the $\mathbb{R}$-sub-vector space $\widetilde{\mathfrak{g}} \subset M_2(\mathbb{C})$ given by

$$\widetilde{\mathfrak{g}} := \{X \in M_2(\mathbb{C}) \mid X + \overline{X}^t = 0, \mathrm{Tr}(X) = 0\} \subset M_2(\mathbb{C}).$$

We have $\dim_{\mathbb{R}}(\widetilde{\mathfrak{g}}) = 3$. We leave as an exercise to check that given $g \in \widetilde{G}$ and $X \in \widetilde{\mathfrak{g}}$, one has $gXg^{-1} \in \widetilde{\mathfrak{g}}$. Therefore, we can think of $\widetilde{\mathfrak{g}}$ as a finite-dimensional $\widetilde{G}$-representation over $\mathbb{R}$ in this way. We leave as an exercise to check that the kernel of the representation is $\{\pm 1\} \subset \widetilde{G}$. As with complex representations, we can average an inner product on the real representation $\widetilde{\mathfrak{g}}$, to obtain a $\widetilde{G}$-invariant inner product. Then the representation map $\widetilde{G} \to \mathrm{GL}_{\mathbb{R}}(\widetilde{\mathfrak{g}})$ will factor via $\widetilde{G} \to O(\widetilde{\mathfrak{g}})$, the subgroup of orthogonal transformations. Since $\widetilde{G}$ is connected, this map in fact is a map $\widetilde{G} \to SO(\widetilde{\mathfrak{g}})$. Thus, we obtain an injective continuous group homomorphism

$$\widetilde{G}/\{\pm 1\} \hookrightarrow SO(\widetilde{\mathfrak{g}}).$$

We leave it as an exercise to check that both groups have the same dimension. From manifold theory, it can be shown that this must be then an open embedding. Since it is also closed as the source is compact, and the target is connected, this must be a homeomorphism. We obtain an isomorphism of compact topological groups

$$SU(2)/\{\pm 1\} \cong SO(3).$$

In particular, we can think of finite-dimensional representations of $SO(3)$ as finite-dimensional representations of $SU(2)$ on which $-1$ acts trivially. Looking at the irreducible finite-dimensional representations $P_n$ above, we see that $-1$ acts trivially on $P_n$ if and only if $n \in 2\mathbb{Z}$. We can conclude that $SO(3)$ has exactly one (up to isomorphism) irreducible representation of dimension $n$ for every $n \in 1 + 2\mathbb{Z}_{\geq 0}$, and those will be all the irreducible finite-dimensional representations (up to isomorphism) of $SO(3)$.

Recall the subgroup $\widetilde{T} \subset \widetilde{G}$ of diagonal matrices, which is isomorphic to $\mathbb{R}/\mathbb{Z}$. We leave as an exercise to see, from linear algebra, that $\widetilde{T}$ will preserve some plane $E \subset \widetilde{\mathfrak{g}}$, and moreover will map isomorphically onto $SO(E)$. In other words, we can choose coordinates so that under the isomorphism $SU(2)/\{\pm 1\} \cong SO(3)$, we will have $\widetilde{T}/\{\pm 1\} \cong SO(2)$ where we consider $SO(2)$ as a subgroup of $SO(3)$ by embedding into the first two coordinates in the standard way. This allows us see that the $(1 + 2n)$-dimensional irreducible representation of $SO(3)$, when decomposed as a $SO(2)$-representation, will have characters $\alpha^{-n}, \alpha^{-n+1}, \ldots, 1, \ldots, \alpha^{n-1}, \alpha^n$, where $\alpha : SO(2) \to \mathbb{C}^\times$ is given by

$$\alpha(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}) = \cos\theta + i \sin\theta.$$

## 8.3 The Peter-Weyl theorem

Throughout this subsection, let us fix a compact topological group $G$. We also denote by $\int$ the normalized Haar measure on $G$.

### 8.3.1

An exercise: Given a compact topological space $X$, and a continuous $G$-action on $X$ (i.e. $G \times X \to X$ is continuous), the induced action of $G$ on $C(X)$ is continuous, where $C(X)$ is considered with the supremum norm. The action is $(gf)(x) := f(g^{-1}x)$.

### 8.3.2

Let $V$ be a finite-dimensional $G$-representation, with action map $\pi : G \to \mathrm{GL}(V)$. We define the **matrix coefficients** map $m_V : \mathrm{End}_{\mathbb{C}}(V) \to C(G)$ by sending $T$ to the function on $G$ sending $g$ to $\mathrm{Tr}(T \circ \pi(g))$. For example, one has $m_V(\mathrm{Id}_V) = \mathrm{ch}_V$. In terms of the isomorphism $V \otimes V^* \cong \mathrm{End}_{\mathbb{C}}(V)$, the matrix coefficients map is the map $V \otimes V^* \to C(G)$ given by sending $v \otimes \alpha$ to the function on $G$ sending $g$ to $\alpha(gv)$.

### 8.3.3

Let $V$ be a finite-dimensional $G$-representation, with action $G \to \mathrm{GL}(V)$. Recall that we have an action of $G \times G$ on $\mathrm{End}_{\mathbb{C}}(V)$, given by

$$(g_1, g_2)T := \pi(g_2) \circ T \circ \pi(g_1)^{-1}.$$

Also, recall that we have an action of $G \times G$ on $C_c(G)$, given by

$$((g_1, g_2)f)(g) := f(g_1^{-1} g g_2).$$

One checks that the matrix coefficients map $m_V : \mathrm{End}_{\mathbb{C}}(V) \to C(G)$ is a $(G \times G)$-morphism.

### 8.3.4

**Lemma 8.30.** *Let $f \in C(G)$. The following conditions are equivalent:*

1. *The function $f$ is in the image of some matrix coefficients map.*
2. *The function $f$ lies in a finite-dimensional $(G \times G)$-subrepresentation of $C(G)$.*
3. *The function $f$ lies in a finite-dimensional $G$-subrepresentation of $C(G)$ with respect to the left $G$-action.*
4. *The function $f$ lies in a finite-dimensional $G$-subrepresentation of $C(G)$ with respect to the right $G$-action.*

*Moreover, the subset $C(G)^{fin} \subset C(G)$ of functions satisfying these conditions is a subspace, closed under pointwise addition, pointwise multiplication and pointwise complex conjugation.*

*Proof.* That (1) implies (2) follows immediately from $m_V$ being $(G \times G)$-morphism, so its image being a finite-dimensional $(G \times G)$-subrepresentation of $C(G)$. It is also clear that (2) implies (3) and (4). Let us see that (4) implies (1) (that (3) implies (1) is analogous). So, let $f \in C(G)$ lie in a finite-dimensional $G$-subrepresentation $V \subset C(G)$ with respect to the left $G$-action. Consider the functional $\ell \in V^*$ given by $\ell(f) := f(1)$. Consider the matrix coefficients map $V \otimes V^* \to C(G)$. One immediately sees that $f$ is the image of $f \otimes \ell$.

□

**Definition 8.31.** The subspace $C(G)^{fin} \subset C(G)$ of functions satisfying the equivalent conditions of the lemma is called the **subspace of finite functions**).

Let $(E_i)_{i \in I}$ be an exhaustive family of irreducible finite-dimensional $G$-representations. We consider the map

$$m := \oplus_{i \in I} m_{E_i} : \bigoplus_{i \in I} \mathrm{End}_{\mathbb{C}}(E_i) \to C(G).$$

**Lemma 8.32.** *The map $m$ is injective, and its image is $C(G)^{fin}$.*

*Proof.* The image is $C(G)^{fin}$ since it is easy to see that a matrix coefficient of a representation can be written as a linear combination of matrix coefficients of irreducible representations. Once we know that each $m_{E_i}$ is injective separately, the map $m$ will be injective by linear independence of isotypic components (we leave the reader to see this). To see that $m_E$ is injective, we can proceed as follows. One sees that $\mathrm{End}_{\mathbb{C}}(E)$ is an irreducible $(G \times G)$-representation (for example by identifying $\mathrm{End}_{\mathbb{C}}(E) \cong E \otimes E^*$ and again performing the exercise that if $E_1$ (resp. $E_2$) is an irreducible finite-dimensional representation of $G_1$ (resp. $G_2$) then $E_1 \otimes E_2$ is an irreducible finite-dimensional representation of $G_1 \times G_2$). Therefore by Schur's lemma either $m_{E_i}$ is equal to zero or is injective. It is not equal to zero since it sends the identity endomorphism to the character, whose inner product with itself is 1. □

**Exercise 8.7.** *Let $E$ be a finite-dimensional $G$-representation. Show that the composition*

$$\mathrm{End}_{\mathbb{C}}(E) \xrightarrow{m_E} C(G) \to \mathrm{End}_{\mathbb{C}}(E),$$

*where the second map is given by sending $f \in C(G)$ to the endomorphism of $E$ sending $v$ to $\int (g \mapsto f(g) \cdot gv)$, is a scalar multiple of the identity. Also, find that scalar.*

### 8.3.5

**Theorem 8.33** (Peter-Weyl)**.** *The subspace $C(G)^{fin}$ is dense in $C(G)$.*

Let us recall the following theorem:

**Theorem 8.34** (Stone-Weierstrass)**.** *Let $X$ be a compact topological space. Let $V \subset C(X)$ be a subspace closed under pointwise sum, product and complex conjugation, and containing $1$. Suppose that $V$ separates points of $X$, meaning that given $x_1, x_2 \in X$ with $x_1 \neq x_2$, there exists $f \in V$ such that $f(x_1) \neq f(x_2)$. Then $V$ is dense in $C(X)$ (where $C(X)$ is equipped with the topology induced by the supremum norm).*

Therefore, to deduce the Peter-Weyl theorem, it is enough to check that $C(G)^{fin}$ separates points of $G$. It is enough to check that for every $g \in G$ such that $g \neq 1$, there exists a finite-dimensional $G$-representation $V$ such that $g$ does not act on $V$ by identity. Indeed, then given $g_1, g_2 \in G$ with $g_1 \neq g_2$, we consider $g_1 g_2^{-1}$ and a finite-dimensional $G$-representation $V$ on which it does not act by identity. Then $g_1$ and $g_2$ do not act identically on $V$ and thus, since one checks that the symmetric bilinear form $(T_1, T_2) \mapsto \mathrm{Tr}(T_1 \circ T_2)$ on $\mathrm{End}_{\mathbb{C}}(V)$ is non-degenerate, there exists $T \in \mathrm{End}_{\mathbb{C}}(V)$ such that $\mathrm{Tr}(g_1 \circ T) \neq \mathrm{Tr}(g_2 \circ T)$, i.e. $m_V(T)(g_1) \neq m_V(T)(g_2)$.

### 8.3.6

Recall that a metric space $(K, d)$ is called **complete** if for a sequence $(x_n)_{n \in \mathbb{Z}_{\geq 1}}$ in $K$, if it is a Cauchy sequence (i.e. for every $\epsilon$ there exists $n_0$ such that $d(x_n, x_m) < \epsilon$ for $n, m > n_0$) then it is convergent (i.e. there exists $x \in K$ such that for every $\epsilon$ there exists $n_0$ such that $d(x_n, x) < \epsilon$ for all $n > n_0$).

In particular, we can talk about complex normed vector spaces, since the norm defines a metric. It is a basic fact that finite-dimensional normed space is always complete. Given a normed vector space $V$, we can form its **completion**, which is a complete normed vector space $V^{\wedge}$ with an isometric embedding $V \to V^{\wedge}$ whose image is dense in $V^{\wedge}$. This in fact characterizes $V^{\wedge}$ completely (one should formulate this in a way similar to the way we formulated the uniqueness of tensor product). One can construct $V^{\wedge}$ as a quotient space of the space of Cauchy sequences in $V$, modulo the subspace of sequences converging to 0. We will not expand the details here.

The completion $V^{\vee}$has the following basic property (which is a characterization of completion when appropriately stated). Let $W$ be a complete normed vector space. Let $T : V \to W$ be a continuous linear map. Then there exists a unique continuous linear map $V^{\vee} \to W$ extending $T$ (which one, by abuse of notation, will also usually denote by $T$).

We want also to recall that given normed vector spaces $V$ and $W$, and a linear map $T : V \to W$, $T$ is continuous if and only if the subset $\{||T(v)|| : v \in V, ||v|| = 1\} \subset \mathbb{R}_{\geq 0}$ is bounded. One also obtains a norm on the space $B(V, W)$ of continuous linear maps from $V$ to $W$, given by $||T|| := sup_{v \in V, ||v||=1} ||T(v)||$.

### 8.3.7

In particular, a vector space $V$ equipped with an inner product $\langle -, - \rangle$ is normed via $||v|| := \sqrt{\langle v, v \rangle}$. This in particular gives $V$ a topology. The inner product $\langle -, - \rangle : V \times V \to \mathbb{C}$ is continuous. One sees that the inner product extends uniquely to a continuous inner product on $V^\vee$, the norm determined by which coincides with the norm already considered on $V^\vee$.

**Definition 8.35.** A **Hilbert space** is a complete inner product space.

A basic construction is the following. Let $X$ be a compact topological space and $\mu$ a nowhere-vanishing Radon measure on $X$. We define an inner product on $C(X)$ by $\langle f_1, f_2 \rangle := \mu(x \mapsto f_1(x)\overline{f_2(x)})$. The completion of $C(X)$ with respect to this inner product is denoted $L^2(X, \mu)$ (or simply $L^2(X)$ if $\mu$ is understood from the context). It is a Hilbert space.

### 8.3.8

**Definition 8.36.** A continuous representation of a locally compact topological group $G$ on a Hilbert space $\mathcal{H}$ is **unitary** if $\langle gv_1, gv_2 \rangle = \langle v_1, v_2 \rangle$ for all $v_1, v_2 \in \mathcal{H}$.

**Remark 8.37.** It can be shown that, in the notation of the definition, given an abstract representation of $G$ on $\mathcal{H}$ for which the map $G \to \mathcal{H}$ given by $g \mapsto gv$ is continuous for every $v \in \mathcal{H}$, and which satisfies the unitarity property, is a continuous representation.

### 8.3.9

We will now concentrate on $L^2(G) := L^2(G, \int)$.

**Lemma 8.38.** *Let $f \in C(G)$. We have $||f||_2 \leq ||f||_{sup}$.*

*Proof.* We have

$$||f||_2^2 = \int (g \mapsto |f(g)|^2) \leq \int (g \mapsto ||f||_{sup}^2) = ||f||_{sup}^2.$$

□

Let $g \in G$. Then the left action of $g$ on $C(G)$ is unitary with respect to the inner product, hence extends to an action on $L^2(G)$. In this way we obtain a unitary action of $G$ on $L^2(G)$. From the above lemma we also easily deduce that this action is continuous. In other words, $L^2(G)$ becomes a unitary $G$-representation.

Let us denote by $L^2(G)^{fin} \subset L^2(G)$ the subspace consisting of elements $f$ which sit in a finite-dimensional $G$-subrepresentation. Let us see now that in order too prove the Peter-Weyl theorem, it is enough to see that $L^2(G)^{fin}$ is dense in $L^2(G)$. Indeed, suppose that this is so. Let $g \in G$ and assume $g \neq 1$.

Then it is easy to see that $g$ does not act as identity on $C(G)$. Therefore, $g$ does not act as identity on $L^2(G)$. Therefore, by our current assumption, $g$ does not act as identity on $L^2(G)^{fin}$. So there exists $f \in L^2(G)^{fin}$ such that $gf \neq f$. Taking a finite-dimensional $G$-subrepresentation $V$ of $L^2(G)$ which contains $f$, we get that $g$ does not act as identity on $V$.

### 8.3.10

**Definition 8.39.** Let $\mathcal{H}$ be a Hilbert space. An operator $T \in B(\mathcal{H})$ is called **compact** if it can be approximated by operators of finite rank. Here, an operator is said to have finite rank if its image is finite-dimensional, and the approximation property means that for every $\epsilon > 0$ there exists an operator of finite rank $S \in B(\mathcal{H})$ such that $||T - S|| < \epsilon$.

**Definition 8.40.** Let $\mathcal{H}$ be a Hilbert space. An operator $T \in B(\mathcal{H})$ is called **self-adjoint** if $\langle T(v_1), v_2 \rangle = \langle v_1, T(v_2) \rangle$ for all $v_1, v_2 \in \mathcal{H}$.

**Theorem 8.41** (Part of spectral theory of compact self adjoint operators)**.** *Let $\mathcal{H}$ be a Hilbert space and $T \in B(\mathcal{H})$ a self-adjoint compact operator. Then the sum of all eigenspaces of $T$ is dense in $\mathcal{H}$ (from this, one sees that the sum of all eigenspaces of $T$ with non-zero eigenvalue is dense in the image of $T$). Moreover, any eigenspace of $T$ with non-zero eigenvalue is finite-dimensional.*

### 8.3.11

Let $k \in C(G \times G)$. Define an operator

$$T_k : C(G) \to C(G)$$

by

$$T_k(f)(g) := \int (x \mapsto k(g, x) f(x)).$$

**Lemma 8.42.** *We have*

$$||T_k(f)||_2 \leq ||k||_{sup} \cdot ||f||_2.$$

*Proof.* Recall that we have $|\int f|^2 \leq \int |f|^2$. Indeed, using Cauchy-Schwartz inequality:

$$|\int f|^2 = |\langle f, 1 \rangle|^2 \leq ||f||_2^2 \cdot ||1||_2^2 = ||f||_2^2 = \int |f|^2.$$

Now, we have:

$$||T_k(f)||_2^2 = \int_x |T_k(f)(x)|^2 = \int_x |\int_y k(x, y) f(y)|^2 \leq \int_x \int_y |k(x, y)| |f(y)|^2 \leq ||k||_{sup}^2 \cdot ||f||_2^2.$$

□

From the lemma, we deduce that

$$C(G) \xrightarrow{T_k} C(G) \hookrightarrow L^2(G)$$

is continuous with respect to the inner product norms, and therefore it extends uniquely to a continuous operator

$$L^2(G) \xrightarrow{T_k} L^2(G),$$

and $||T_k|| \leq ||k||_{sup}$.

**Lemma 8.43.** *The operator*

$$L^2(G) \xrightarrow{T_k} L^2(G)$$

*is compact.*

*Proof.* Consider in $C(G \times G)$ the subspace spanned by functions of the form $(g_1, g_2) \mapsto f_1(g_1)f_2(g_2)$ where $f_1, f_2 \in C(G)$. By the Stone-Weierstrass theorem, this subspace is dense in $C(G \times G)$. Let therefore $k_n$ be a sequence of functions in the subspace converging to $k$. We have $||T_k - T_{k_n}|| = ||T_{k-k_n}|| \leq ||k - k_n||_{sup}$. Furthermore, notice that each $T_{k_n}$ has an at most 1-dimensional image, so is of finite rank. □

**Lemma 8.44.** *The operator $T_k$ is adjoint to the operator $T_{\overline{k}^*}$, where $\overline{k}^*(x, y) := \overline{k(y, x)}$. In particular, if $k = \overline{k}^*$ then $T_k$ is self-adjoint.*

*Proof.* Left as an exercise for now. □

Let now $f \in C(G)$. Define $k_f(x, y) := f(xy^{-1})$.

**Definition 8.45.** Let $U$ be an open neighbourhood of 1 in $G$. A **$U$-unit approximate** is a function $h \in C(G)$ whose support lies in $U$, whose values lie in $\mathbb{R}_{\geq 0}$, and such that $\int h = 1$.

**Lemma 8.46.** *Let $f \in C(G)$. For every $\epsilon > 0$ there exists an open neighbourhood $U$ of 1 in $G$ such that for every $U$-unit approximate $h \in C(G)$, we have $||T_{k_h} f - f||_{sup} \leq \epsilon$.*

*Proof.* From basic topology, there exists an open neighbourhood $U$ of 1 in $G$ such that $|f(u^{-1}g) - f(g)| \leq \epsilon$ for all $g \in G$ and $u \in U$. Let $h$ be a $U$-unit approximate. Then for any $g \in G$ we have

$$|(T_{k_h}(f) - f)(g)| = |\int (x \mapsto h(gx^{-1})f(x)) - f(g)| = |\int (x \mapsto h(gx^{-1})(f(x) - f(g)))| \leq$$

$$\leq \int (x \mapsto h(gx^{-1})|f(x) - f(g)|) \leq \epsilon.$$

□

**Lemma 8.47.** *Let $U$ be an open neighbourhood of $1$ in $G$. Then there exists a $U$-unit approximate $h$ such that $h \in C(G)^{cl}$ and $h = h^*$.*

*Proof.* Denote $U_1 := U \cap U^{-1}$. Since $G$ is compact, from basic topology we can find an open neighbourhood $V$ of $1$ in $G$ such that for $x \in V$ and $g \in G$ we have $gxg^{-1} \in U_1$. Let $h_0 \in C(G)$ be a $V$-unit approximate. Denote by $h_1 \in C(G)$ the function $h_1(x) := \int (g \mapsto h_0(g^{-1}xg))$. Then $h_1 \in C(G)^{cl}$ and $h_1$ is a $U_1$-unit approximate. Now set $h \in C(G)$ to be $h(x) := \frac{1}{2}(h_1(x) + h_1(x^{-1}))$. Then $h \in C(G)^{cl}$, $h = h^*$ and $h$ is a $U$-unit approximate. □

### 8.3.12

We can now finish the proof of the Peter-Weyl theorem. We want to show that $L^2(G)^{fin}$ is dense in $L^2(G)$. Since $C(G)$ is dense in $L^2(G)$, it is enough to take $f \in C(G)$ and show that it can be approximated by elements of $L^2(G)^{fin}$. Let $\epsilon > 0$. We saw above that $f$ can be approximated to any degree by elements $T_{k_h} f$ in the supremum norm, and therefore in the $L^2$-norm, for $U$-unit approximates $h$. By above, we can choose to take $h \in C(G)^{cl}$ and $h = h^*$. Then by the above $T_{k_h}$ is a compact self-adjoint operator. By the spectral theorem, $T_{k_h} f$ can be approximated to any degree by finite sums of eigenvectors of $T_{k_h}$ with non-zero eigenvalues. These eigenvectors sit in the finite-dimensional eigenspaces. Notice now that since $h \in C(G)^{cl}$, the eigenspaces of $T_{k_h}$ are $G$-subrepresentations of $L^2(G)$, and therefore the eigenspaces with non-zero eigenvalues lie in $L^2(G)^{fin}$.

### 8.3.13

Let us give an application. Let $\alpha \in \mathbb{R}$ be an irrational number. Denote by $[-] : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ the projection. We claim that $\{[n\alpha]\}_{n \in \mathbb{Z}}$ is equi-distributed in $\mathbb{R}/\mathbb{Z}$. This means that for any interval $[c_1, c_2] \subset [0, 1]$, we have:

$$\lim_{N \to \infty} \frac{1}{N+1} |\{0 \leq n \leq N : \ [n\alpha] \in [c_1, c_2]\}| = c_2 - c_1.$$

It is not hard to see that this is equivalent to the following: For any $f \in C(\mathbb{R}/\mathbb{Z})$, we have

$$\lim_{N \to \infty} \sum_{0 \leq n \leq N} f([n\alpha]) = \int f$$

where $\int$ is the normalized Haar measure on $\mathbb{R}/\mathbb{Z}$. It is immediate to see that it is enough to prove this for $f$ lying in a subset of $C(G)$, whose span is dense in $C(G)$ (with respect to the supremum norm topology). By the Peter-Weyl theorem, $\{\chi_m\}_{m \in \mathbb{Z}}$ is such a subset (of course, in this case it is easy to see the separation of points directly, but this is just an illustration). For $\chi_0$ the claim is clear. Let $m \neq 0$. We have $\int \chi_m = \langle \chi_m, \chi_0 \rangle = 0$. Notice that $\chi_m([\alpha]) \neq 1$, because $\alpha$ is not rational. Denote $\beta := \chi_m([\alpha])$. We have:

$$\frac{1}{N+1} \sum_{0 \leq n \leq N} \beta^n = \frac{1}{N+1} \sum_{0 \leq n \leq N} e^{2\pi i (n\alpha)} = \frac{1}{N+1} \cdot \frac{\beta^{N+1} - 1}{\beta - 1} \to 0.$$

**8.3.14**

We now also want to discuss the $L^2$-formulation of the Peter-Weyl theorem.

Let $V$ be a finite-dimensional inner product space. One has an inner product on $\mathrm{End}_{\mathbb{C}}(V)$, known as the **Hilbert-Schmidt** inner product, given by $\langle T, S\rangle := \mathrm{Tr}(T \circ S^*)$. Here $S^* \in \mathrm{End}_{\mathbb{C}}(V)$ is the adjoint to $S$, characterized by $\langle S^*(v), w\rangle = \langle v, S(w)\rangle$ for all $v, w \in V$.

Let $V$ be a finite-dimensional unitary $G$-representation. One checks that the action of $G \times G$ on $\mathrm{End}_{\mathbb{C}}(V)$ is unitary (where the latter is given the Hilbert-Schmidt inner product). Also, one checks that the action of $G \times G$ on $C(G)$ is unitary.

**Exercise 8.8.** *Let $E$ be an irreducible unitary $G$-representation. Show that*

$$\sqrt{\dim E} \cdot m_E : \mathrm{End}_{\mathbb{C}}(E) \to C(G)$$

*is isometric.*

Let $(\mathcal{H}_i)_{i \in I}$ be a collection of Hilbert spaces. We can construct a Hilbert space $\oplus^{\vee}_{i \in I} \mathcal{H}_i$ by first defining an inner product on $\oplus_{i \in I} \mathcal{H}_i$ by

$$\langle \sum_i v_i, \sum_i w_i \rangle = \sum_i \langle v_i, w_i \rangle,$$

and then taking the completion of the resulting inner product space.

**Exercise 8.9.** *Interpret the Peter-Weyl theorem by saying that the maps from Exercise 8.8 induce an isomorphism*

$$\oplus^{\wedge}_{i \in I} \mathrm{End}_{\mathbb{C}}(E) \cong L^2(G)$$

*(where again $(E_i)_{i \in I}$ is an exhaustive family of irreducible finite-dimensional $G$-representations). This is an isomorphism of unitary $G$-representations (i.e. both an isomorphism of Hilbert spaces and a $G$-morphism).*

**8.3.15**

Recall that we denote by $L^2(G)^{fin} \subset L^2(G)$ the subspace of vectors which sit in a finite-dimensional $G$-subrepresentation, with respect to the left $G$-action. Clearly $C(G)^{fin} \subset L^2(G)^{fin}$. We have the following important "automatic regularity"-type statement:

**Proposition 8.48.** *We have $C(G)^{fin} = L^2(G)^{fin}$.*

*Proof.* We want to show that a $G$-finite $f \in L^2(G)$ is automatically continuous. We can write $f$ as a sum of elements which sit in irreducible finite-dimensional $G$-subrepresentations, and therefore without loss of generality we can assume that $f$ itself sits in an irreducible finite-dimensional $G$-subrepresentation $E$. By the Peter-Weyl theorem, we can approximate $f$ to any degree (in the $L^2$-norm) by elements of $C(G)^{fin}$. An element of $C(G)^{fin}$ we also write as a sum of elements sitting in irreducible finite-dimensional subrepresentations. Recall that we saw that elements sitting in non-isomorphic irreducible subrepresentations must be orthogonal. This shows that since our $f$ can be approximated by elements of $C(G)^{fin}$, it can in fact be approximated by elements of $m_E(\mathrm{End}_{\mathbb{C}}(E)) \subset C(G)^{fin}$. Since $m_E(\mathrm{End}_{\mathbb{C}}(E))$ is a finite-dimensional subspace of $L^2(G)$, it is complete and hence closed in $L^2(G)$. Therefore our $f$ must lie in $m_E(\mathrm{End}_{\mathbb{C}}(E))$, and in particular is continuous. □

### 8.3.16

Let $H \subset G$ be a closed subgroup. We consider $Y := G/H$. It is a compact topological space, equipped with a continuous $G$-action $G \times Y \to Y$. In particular, we obtain a $G$-representation $C(Y)$ (with the supremum norm). We have the subspace $C(Y)^{fin}$ consisting of elements which sit in a finite-dimensional $G$-subrepresentation.

Notice that we have an inclusion $C(G/H) \hookrightarrow C(G)$ by pulling back, and in fact this identifies $C(G/H)$ with a the closed subspace $C(G)^H \subset C(G)$, where here $H$-invariants are with respect to the right $H$-action. This is also isometric with respect to the supremum norms. Using that, we get:

**Claim 8.49.** *$C(G/H)^{fin}$ is dense in $C(G/H)$ (with respect to the supremum norm) and one has $C(G/H)^{fin} \cong \oplus_{i\in I} E_i \otimes (E_i^*)^H$.*

*Proof.* We have $C(G/H)^{fin} \cong (C(G)^{fin})^H$. We leave to the reader for now to check the density claim - one uses the density for $C(G)$ (the Peter-Weyl theorem) and averaging on the right with repsect to $H$. Recall the isomorphism of $(G \times G)$-representations

$$C(G)^{fin} \cong \oplus_{i\in I} E_i \otimes E_i^*.$$

We obtain

$$C(G/H)^{fin} \cong (C(G)^{fin})^H \cong \oplus_{i\in I} E_i \otimes (E_i^*)^H.$$

□

One has the analog of Haar's theorem:

**Theorem 8.50** (Haar's theorem)**.** *There exists a $G$-invariant nowhere-vanishing Radon measure $\int$ on $Y$, which can be normalized by requiring $\int 1 = 1$. Any $G$-invariant signed Radon measure on $Y$ differs from that one by a scalar.*

We then set $L^2(Y) := L^2(Y, \int)$ and $L^2(Y)$ is then naturally a unitary $G$-representation. We can again set $L^2(Y)^{fin} \subset L^2(Y)$ to consist of the $G$-finite vectors, and as before we see that $C(Y)^{fin} = L^2(Y)^{fin}$. We obtain

$$L^2(G/H) = \oplus^{\wedge}_{i\in I} E_i \otimes (E_i^*)^H$$

for some suitable inner products on $E_i \otimes (E_i^*)^H$.

We tell this because we want to have the following illustration. We take $G := SO(3)$ and $H = SO(2) \subset SO(3)$ (embedded in the standard way via the first two coordinates). If we let $G$ act on the two-dimensional sphere $S$ in the standard way, the action is transitive and $H$ is the stabilizer of a certain point. Therefore we can identify $G/H \cong S$. One obtains a measure on $S$ - the unique Radon measure of total mass 1 which is preserved under rotations. One obtains the corresponding Hilbert space $L^2(S)$, equipped with a unitary representation of $G$. It is very natural now to find the generalization of what we had for $S^1$ - the eigenbasis of characters (here "basis" in the Hilbert sense). As we know from the experience with finite groups, we can't expect an eigenbasis, but a decomposition into direct sum (in the Hilbert sense) of irreducibles. And indeed, as we explained, we have

$$L^2(S) \cong \oplus^{\wedge}_{n \in 1+2\mathbb{Z}_{\geq 0}} E_n \otimes (E_n^*)^H,$$

where $E_n$ is the unique (up to isomorphism) unitary irreducible representation of $G$ of dimension $n$. Above we saw how the $E_n$'s deompose under the $H$-action. In particular, we see that $\dim(E_n^*)^H = 1$ for all $n$. To summarize, we obtain the following:

**Proposition 8.51** (Part of the theory of spherical harmonics)**.** *For every $n \in 1 + 2\mathbb{Z}_{\geq 0}$, there exists a unique subrepresentation $F_n$ of $L^2(S)$ of dimension $n$. We obtain*

$$L^2(S) = \oplus^{\wedge}_{n \in 1+2\mathbb{Z}_{\geq 0}} F_n.$$

*We have $\oplus_{n \in 1+2\mathbb{Z}_{\geq 0}} F_n \subset C(S)$, and this is a dense inclusion with respect to the supremum norm.*

## 8.4 The Lie algebra approach

Throughout this subsection, let $G$ be a compact topological group, assumed to be a closed subgroup of some $\mathrm{GL}_n(\mathbb{R})$ (such a group is called a **matrix group**). This in fact restricts us to so-called Lie groups (for example, excluding $p$-adic groups).

### 8.4.1

Recall the exponential map $M_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ given by $X \mapsto e^X := \sum_{m \in \mathbb{Z}_{\geq 0}} \frac{1}{n!} X^n$.

**Exercise 8.10.** *Let $\lim_{n\to\infty} X_n = X$. Show that*

$$e^X = \lim_{n\to\infty} \left( I + \frac{1}{n} X_n \right)^n$$

**Lemma 8.52.** *Let $X_1, X_2 \in M_n(\mathbb{R})$. Then*

$$e^{X_1+X_2} = \lim_{n\to\infty} \left( e^{\frac{1}{n}X_1} e^{\frac{1}{n}X_2} \right)^n.$$

*Proof.* We have

$$e^{\frac{1}{n}X_1}e^{\frac{1}{n}X_2} = \left(I + \frac{1}{n}X_1 + O(\frac{1}{n^2})\right) \cdot \left(I + \frac{1}{n}X_2 + O(\frac{1}{n^2})\right) =$$

$$= I + \frac{1}{n}(X_1 + X_2 + O(\frac{1}{n}))$$

and thus

$$\left(e^{\frac{1}{n}X_1}e^{\frac{1}{n}X_2}\right)^n = \left(I + \frac{1}{n}(X_1 + X_2 + O(\frac{1}{n}))\right)^n \xrightarrow[n\to\infty]{} e^{X_1+X_2}.$$

□

We define the **Lie algebra** of $G$ as the subset of $M_n(\mathbb{R})$ consisting of matrices $X$ for which $e^{tX} \in G$ for all $t \in \mathbb{R}$. We denote it by $\mathrm{Lie}(G)$.

**Claim 8.53.** $\mathrm{Lie}(G)$ *is a* $\mathbb{R}$*-vector subspace of* $M_n(\mathbb{R})$*. Moreover, given* $X_1, X_2 \in \mathrm{Lie}(G)$*, we have* $[X_1, X_2] := X_1X_2 - X_2X_1 \in \mathrm{Lie}(G)$*. Also, given* $g \in G$ *and* $X \in \mathrm{Lie}(G)$ *we have* $gXg^{-1} \in \mathrm{Lie}(G)$*.*

*Proof.* It is clear that $\mathrm{Lie}(G)$ is closed under scalar multiplication and that $0 \in \mathrm{Lie}(G)$. Let $X_1, X_2 \in \mathrm{Lie}(G)$. Since $G$ is closed in $\mathrm{GL}_n(\mathbb{R})$, it is clear from the above formula that $e^{t(X_1+X_2)} \in \mathrm{Lie}(G)$ for any $t \in \mathbb{R}$, so $X_1 + X_2 \in \mathrm{Lie}(G)$. Given $g \in G$ and $X \in \mathrm{Lie}(G)$ we have $e^{t \cdot gXg-1} = ge^{tX}g^{-1} \in G$. It is now left to see that if $X_1, X_2 \in \mathrm{Lie}(G)$ then $[X_1, X_2] \in \mathrm{Lie}(G)$. It is standard to calculate that

$$\lim_{t\to 0} \frac{e^{tX_1} - 1}{t} = X_1.$$

Then

$$\lim_{t\to 0} \frac{e^{tX_1}X_2 - X_2}{t} = X_1X_2.$$

Therefore

$$\lim_{t\to 0} \frac{e^{tX_1}X_2e^{-tX_1} - X_2}{t} = \lim_{t\to 0} \frac{e^{tX_1}X_2e^{-tX_1} - e^{tX_1}X_2}{t} + \lim_{t\to 0} \frac{e^{tX_1}X_2 - X_2}{t} =$$

$$= -X_2X_1 + X_1X_2 = [X_1, X_2].$$

□

**Definition 8.54.** A **Lie algebra** (over $\mathbb{R}$) is an $\mathbb{R}$-vector space $\mathfrak{g}$ equipped with a bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying:

1. $[X_2, X_1] = -[X_1, X_2]$.
2. $[X_1, [X_2, X_3]] = [[X_1, X_2], X_3] + [X_2, [X_1, X_3]]$.

**Exercise 8.11.** *Let* $A$ *be a* $\mathbb{R}$*-algebra. Show that* $A$ *equipped with* $[a_1, a_2] := a_1a_2 - a_2a_1$ *is a Lie algebra.*

We have the notion of Lie subalgebra which we leave to the reader to define. We see that $\mathrm{Lie}(G)$ is a Lie algebra, a Lie subalgebra of $M_n(\mathbb{R})$ equipped with the bracket $[X_1, X_2] := X_1X_2 - X_2X_1$.

**Remark 8.55.** To have a description of the Lie algebra $\mathrm{Lie}(G)$ that is manifestly independent of the choice of embedding into $\mathrm{GL}_n(\mathbb{R})$, one has to talk about smooth manifolds and Lie groups. Then given a Lie group $G$, one can define an intrinsic $\mathbb{R}$-vector space, the tangent space to $G$ at $1 \in G$, and on it a Lie algebra structure. Then our definition acquires its "correct" role, as a calculation.

### 8.4.2

Let us calculate the Lie algebra of $G := SU(n)$. We will consider $G$ as a closed subgroup in $M_n(\mathbb{C})$ in the natural way, which in its turn we will consider as a closed subgroup in $M_{2n}(\mathbb{R})$. Then $\mathrm{Lie}(G)$ is defined as a subspace of $M_{2n}(\mathbb{R})$. However, since $X \in \mathrm{Lie}(G)$ can be written as $\lim_{t\to 0} \frac{e^{tX}-I}{t}$, we see that $\mathrm{Lie}(G) \subset M_n(\mathbb{C})$. So we need to find matrices $X \in M_n(\mathbb{C})$ for which $e^{tX} \in SU(n)$ for all $t \in \mathbb{R}$.

**Exercise 8.12.** *Show that*

$$\mathrm{Lie}(SU(n)) = \{X \in M_n(\mathbb{C}) \mid X + \overline{X}^t = 0, \ \mathrm{Tr}(X) = 0\}.$$

### 8.4.3

Let $V$ be a finite-dimensional $G$-representation. It can be shown that for any $X \in \mathrm{Lie}(G)$ and $v \in V$, the limit $X \cdot v := \lim_{t\to 0} \frac{1}{t}(e^{tX}v - v)$ exists (this is some result of "automatic smoothness" type, that a continuous finite-dimensional representation is automatically smooth). In this way we obtain a map $\mathrm{Lie}(G) \times V \to V$.

**Lemma 8.56.** *In the above notation, the map* $\mathrm{Lie}(G) \times V \to V$ *is* $\mathbb{R}$*-linear in the first variable,* $\mathbb{C}$*-linear in the second variable, and we have* $[X_1, X_2] \cdot v = X_1 \cdot (X_2 \cdot v) - X_2 \cdot (X_1 \cdot v)$ *for all* $X_1, X_2 \in \mathrm{Lie}(G)$ *and* $v \in V$*.*

*Proof.* It is clear that the map is complex linear in the second variable. Respecting multiplication by real scalar in the first variable is also easy. Let us show additivity in the first variable: complete □

**Definition 8.57.** Let $\mathfrak{g}$ be a $\mathbb{R}$-Lie algebra. Let $V$ be a $\mathbb{C}$-vector space. A **representation** of $\mathfrak{g}$ on $V$ (or the structure of a $\mathfrak{g}$**-module** on $V$) is a map $\mathfrak{g} \times V \to V$, $\mathbb{R}$-linear in the first variable, $\mathbb{C}$-linear in the second variable, and such that $[X_1, X_2] \cdot v = X_1 \cdot (X_2 \cdot v) - X_2 \cdot (X_1 \cdot v)$ for all $X_1, X_2 \in \mathfrak{g}$ and $v \in V$. We leave to the reader to define morphisms etc.

**Lemma 8.58.** *Let* $V_1, V_2$ *be finite-dimensional* $G$*-representations, and let* $T : V_1 \to V_2$ *be a* $G$*-morphism. Then* $T$ *is also a* $\mathrm{Lie}(G)$*-morphism.*

*Proof.* Left as an exercise. □

#### 8.4.4

The following are parts of what is known as Lie's theorems:

**Theorem 8.59.** *Suppose that $G$ is connected. Let $V$ and $W$ be finite-dimensional $G$-representations. Let $T : V \to W$ be a linear map. If $T$ is a* $\mathrm{Lie}(G)$*-morphism then $T$ is a $G$-morphism.*

**Theorem 8.60.** *Suppose that $G$ is connected and simply connected. Let $V$ be a finite-dimensional module over* $\mathrm{Lie}(G)$*. Then there exists a continuous $G$-action on $V$ for which the induced* $\mathrm{Lie}(G)$*-action is the given one.*

**Corollary 8.61.** *We see that if $G$ is connected and simply connected, finite-dimensional $G$-representations are the same as finite-dimensional* $\mathrm{Lie}(G)$*-modules. In this way, representation theory of compact matrix groups is algebraic!*

**Remark 8.62.** Let $\mathfrak{g}$ be a $\mathbb{R}$-Lie algebra. We can consider the complexification $\mathfrak{g}_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$, which is a $\mathbb{C}$-Lie algebra naturally. Then $\mathfrak{g}$-modules are the same as $\mathfrak{g}_{\mathbb{C}}$-modules.

Let us now consider $G := SU(n)$. We claim that $G$ is connected and simply connected. Recall that

$$\mathrm{Lie}(G) = \{X \in M_n(\mathbb{C}) \mid X + \overline{X}^t = 0, \mathrm{Tr}(X) = 0\} \subset M_n(\mathbb{C}).$$

We have the corresponding emebedding of $\mathbb{C}$-Lie algebras. $\mathrm{Lie}(G)_{\mathbb{C}} \hookrightarrow M_n(\mathbb{C})$. Clearly the image lies in the Lie-subalgebra

$$\mathfrak{sl}_n := \{X \in M_n(\mathbb{C}) \mid \mathrm{Tr}(X) = 0\}.$$

By dimension consideration, we have an equality:

$$\mathrm{Lie}(G)_{\mathbb{C}} = \mathfrak{sl}_n.$$

**Corollary 8.63.** *The study of finite-dimensional $SU(n)$-representations is the same as the study of finite-dimensional modules over the complex Lie algebra $\mathfrak{sl}_n$!*

### 8.5 Representation theory of $\mathfrak{sl}_n$

We are interested in understanding the finite-dimensional irreducible $\mathfrak{sl}_n$-modules. Let us abbreviate $\mathfrak{g} := \mathfrak{sl}_n$ and we denote by $\mathfrak{t}$ the Lie subalgebra of $\mathfrak{g}$ consisting of the diagonal matrices.

#### 8.5.1

We have the following result:

**Proposition 8.64.** *Let $V$ be a finite-dimensional $\mathfrak{sl}_n$-module. Then $V$ is semisimple - given a submodule $W \subset V$, there exists a submodule $U \subset V$ such that $V = W \oplus U$.*

*Proof.* This follows from Lie's theorems, since finite-dimensional $\mathfrak{sl}_n$-modules are precisely the same as finite-dimensional $SU(n)$-representations. $\square$

**Remark 8.65.** One can also prove the proposition in a completely algebraic way, without referring to the connection to $SU(n)$.

### 8.5.2

In a similar spirit, we have the following:

**Proposition 8.66.** *Let $V$ be a finite-dimensional $\mathfrak{sl}_n$-module. Then $V$ is $\mathfrak{t}$-diagnolizable, i.e. $V$ is the direct sum of subspaces $V_\lambda$, where $\lambda \in \mathfrak{t}^*$ and*

$$V_\lambda := \{v \in V \mid Hv = \lambda(H)v \ \forall H \in \mathfrak{t}\}.$$

*Proof.* We can treat $V$ as a $SU(n)$-representation. As before, we denote by $T \subset SU(n)$ the subgroup consisting of diagonal matrices. Then $V$ is the direct sum of $T$-isotypic components $V_{T,\chi}$. By definitions, given $v \in V_{T,\chi}$ and $H := \mathrm{diag}(it_1, \dots, it_n) \in \mathfrak{t}$, we have

$$H \cdot v = \lim_{t \to 0} \frac{1}{t} \left( \mathrm{diag}(e^{itt_1}, \dots, e^{itt_n})v - v \right) = \left( \lim_{t \to 0} \frac{\chi(\mathrm{diag}(e^{itt_1}, \dots, e^{itt_n})) - 1}{t} \right) \cdot v.$$

We define $d\chi \in \mathfrak{t}^*$ by $(d\chi)(\mathrm{diag}(t_1, \dots, t_n)) := \lim_{t \to 0} \frac{\chi(\mathrm{diag}(e^{tt_1}, \dots, e^{tt_n})) - 1}{t}$. Then $\mathfrak{t}$ acts on $V_{T,\chi}$ via $d\chi$. $\square$

**Remark 8.67.** Similarly to the previous remark, one can prove this proposition completely algebraically, without referring to the connection to $SU(n)$.

### 8.5.3

Since finite-dimensional $\mathfrak{sl}_2$-modules are the same as finite-dimensional $SU(2)$-representations, we have a knowledge of irreducible finite-dimensional $\mathfrak{sl}_2$-module, which we now describe. For any $m \in \mathbb{Z}_{\geq 0}$, there is exactly one, up to isomorphism, finite-dimensional irreducible $\mathfrak{sl}_2$-module $P_m$ of dimension $m+1$. We can also calculate how $P_m$ looks like as a $\mathfrak{t}$-module: consider the basis element $H_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{t}$. By the definitions, the action of $iH_0$ on $P_m$ is given as $\lim_{t \to 0} \frac{1}{t}(e^{t \cdot iH_0} - 1)$. Recall that $P_n$ decomposes into one-dimensional $T$-eigenspaces, where $T$ is the subgroup of diagonal elements in $SU(2)$, corresponding to characters $\chi_d$ of $T$, for $d \in \{-m, -m+2, \dots, m-2, m\}$ where $\chi_d(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}) = e^{i \cdot d\theta}$. Therefore, we conclude that the action of $H_0$ on $P_m$ is diagnolizable, with one-dimensional eigenspaces, with eigenvalues running over $\{-m, -m+2, \dots, m-2, m\}$.

**Remark 8.68.** Similarly to the previous remarks, one can obtain this information completely algebraically, without referring to the connection to $SU(2)$.

### 8.5.4

To go further in trying to somehow parametrize irreducible finite-dimensional $\mathfrak{sl}_n$-representations, we need some more language to talk about functionals in $\mathfrak{t}^*$ (which are parametrizing the $\mathfrak{t}$-eigenspaces in finite-dimensional $\mathfrak{sl}_n$-modules). A usual terminology is to call elements of $\mathfrak{t}^*$ **weights**, to call $\mathfrak{t}$-eigenspaces **weight spaces**, and to call vectors in weight spaces **weight vectors**. We can say that a weight $\lambda \in \mathfrak{t}^*$ **appears** in a $\mathfrak{g}$-module $V$, or is a **weight of** $V$ if the weight space $V_{\mathfrak{t},\lambda}$ is non-zero, where

$$V_{\mathfrak{t},\lambda} := \{v \in V \mid H \cdot v = \lambda(H) \cdot v, \ \forall H \in \mathfrak{t}\}.$$

Similarly to the regular representation of a group, every Lie algebra $\mathfrak{g}$ is a module over itself, by $X \cdot Y := [X, Y]$. We want to decompose our $\mathfrak{g} := \mathfrak{sl}_n$ into weight spaces. For a pair $(i, j)$ of indices in $[1, n]$, such that $i \neq j$, one has the element $E_{i,j} \in \mathfrak{g}$ whose $(i, j)$-entry is 1 and all other entries are 0. Writing, given $H \in \mathfrak{t}$, $H = \mathrm{diag}(H^{(1)}, \ldots, H^{(n)})$, we have

$$[H, E_{i,j}] = (H^{(i)} - H^{(j)}) \cdot E_{i,j}.$$

Thus, we have $(n-1)$-dimensional weight space $\mathfrak{t} \subset \mathfrak{g}$ corresponding to weight 0, and we have 1-dimensional weight spaces, corresponding to weights $\alpha_{i,j}(H) := H^{(i)} - H^{(j)}$ where $(i, j) \in [1, n]^2$ is such that $i \neq j$. A usual terminology is to call the non-zero weights appearing in the regular $\mathfrak{g}$-module $\mathfrak{g}$ **roots**. Thus, the roots in our case are the elements in $\mathfrak{t}^*$ of the form $\alpha_{i,j}$ with $(i, j) \in [1, n]^2$ and $i \neq j$. The elements $\alpha_{i,j}$ with $j > i$ we call **positive roots**, and the elements $\alpha_{i,j}$ with $j < i$ we call **negative roots**. For a root $\alpha := \alpha_{i,j}$ let us write also $E_\alpha := E_{i,j}$. Notice that $\alpha_{i,j} = -\alpha_{j,i}$, i.e. negative roots are simply negatives of positive roots.

### 8.5.5

Let us gather a few tools for the continuation of the discussion. We need to understand a weak version of the Poincare-Birkhoff-Witt theorem.

**Proposition 8.69.** *Let $V$ be a $\mathfrak{g}$-module and let $X_1, \ldots, X_d$ be a basis for $\mathfrak{g}$. Let $v \in V$. Then the subspace $W \subset V$ spanned by elements $X_d^{m_d} \cdot \ldots \cdot X_1^{m_1} v$, for $m_i \in \mathbb{Z}_{\geq 0}$, is a $\mathfrak{g}$-submodule of $V$, and is contained in any $\mathfrak{g}$-submodule of $V$ containing $v$.*

Also, we have the following important computation:

**Lemma 8.70.** *Let $V$ be a $\mathfrak{g}$-module. Let $v \in V_{\mathfrak{t},\lambda}$. Let $E \in \mathfrak{g}_{\mathfrak{t},\omega}$. Then $E \cdot v \in V_{\mathfrak{t},\omega+\lambda}$.*

Finally, let us define a partial order on $\mathfrak{t}^*$, by setting $\lambda_1 \leq \lambda_2$ if $\lambda_2 - \lambda_1$ can be written as a finite (possibly empty) sum of positive roots.

### 8.5.6

**Definition 8.71.** Let $V$ be a $\mathfrak{g}$-module. A non-zero $v \in V$ is called a **highest weight vector** if $v$ is a weight vector and $E_\alpha \cdot v = 0$ for all positive roots $\alpha$. A non-zero vector $v \in V$ is called a **highest weight generator** if it is a highest weight vector and it is a generator of $V$ as a $\mathfrak{g}$-module (the latter means that any $\mathfrak{g}$-submodule of $V$ containing $v$ is equal to $V$). The module $V$ is called a **highest weight module** if it admits a highest weight generator.

**Proposition 8.72.** *Let $V$ be an irreducible finite-dimensional $\mathfrak{g}$-module. Then $V$ is a highest weight module.*

*Proof.* In the finite set of weights of $V$, let us pick a weight $\lambda \in \mathfrak{t}^*$, maximal with respect to our partial order. Let $0 \neq v \in V_{\mathfrak{t},\lambda}$. We first claim that $v$ is a highest weight vector. Indeed, let $\alpha$ be a positive root. Then $E_\alpha \cdot v \in V_{\alpha+\lambda}$. But by the maximality assumption on $\lambda$ we have that $\alpha+\lambda$ is not a weight of $V$, i.e. $V_{\alpha+\lambda} = 0$. Thus $E_\alpha \cdot v = 0$. So $v$ is a highest weight vector. Furthermore, clearly the submodule of $V$ generated by $v$ is the whole $V$, since $V$ is irreducible. □

**Exercise 8.13.** *Using the semisimplicity of finite-dimensional $\mathfrak{g}$-modules, see that the converse also holds: a finite-dimensional highest weight $\mathfrak{g}$-module is necessarily irreducible.*

We now study highest weight modules, and in particular irreducible highest weight modules, in more detail.

**Proposition 8.73.** *Let $V$ be a highest weight $\mathfrak{g}$-module with highest weight generator $v$, having weight $\lambda$. Then all weights $\mu$ of $V$ satisfy $\mu \leq \lambda$. Also, we have $\dim V^{\mathfrak{t},\lambda} = 1$. The vector $v$ is the unique highest weight generator of $V$, up to scalar. Finally, $V$ is an indecomposable $\mathfrak{g}$-module.*

*Proof.* Let us enumerate the positive roots by $\alpha_1, \ldots, \alpha_d$, and pick a basis $X_1, \ldots, X_d, H_1, \ldots, H_{n-1}, Y_1, \ldots, Y_d$ of $\mathfrak{g}$ where $X_i \in \mathfrak{g}_{\mathfrak{t},\alpha_i}$, $H_1, \ldots, H_{n-1}$ is a basis of $\mathfrak{t}$, and $Y_i \in \mathfrak{g}_{\mathfrak{t},-\alpha_i}$. By above, the subspace of $V$ spanned by elements $Y_d^? \ldots Y_1^? H_{n-1}^? \ldots H_1^? X_d^? \ldots X_1^? v$ is a $\mathfrak{g}$-submodule. Since $V$ is generated by $v$ as a $\mathfrak{g}$-module, this subspace must be equal to the whole $V$. Note that each $Y_d^? \ldots Y_1^? H_{n-1}^? \ldots H_1^? X_d^? \ldots X_1^? v$ is a scalar multiple of $Y_d^? \ldots Y_1^? v$, so

$$Y_d^{m_d} \ldots Y_1^{m_1} H_{n-1}^? \ldots H_1^? X_d^? \ldots X_1^? v \in V_{\mathfrak{t},\lambda-(m_1\alpha_1+\ldots+m_d\alpha_d)}.$$

Therefore, the weights of $V$ lie in the set of weights of the form $\lambda-(m_1\alpha_1+\ldots+m_d\alpha_d)$ for $(m_1, \ldots, m_d) \in \mathbb{Z}_{\geq 0}^d$. In particular, all are $\leq \lambda$. Also, we see from here that $\dim V^{\mathfrak{t},\lambda} = 1$. It is clear that $v$ is the unique highest weight generator of $V$, because if we have another highest weight generator, with weight $\mu$, then by what we have seen just now we have $\lambda \leq \mu$ and $\mu \leq \lambda$ and therefore $\mu = \lambda$ and so our second generator lies in $V^{\mathfrak{t},\lambda}$, and so is a scalar multiple of $v$. Finally, to see that $V$ is indecomposable, suppose that $V = V_1 \oplus V_2$. Then $V^{\mathfrak{t},\lambda} = V_1^{\mathfrak{t},\lambda} \oplus V_2^{\mathfrak{t},\lambda}$. Since $V^{\mathfrak{t},\lambda}$ is 1-dimensional, we have $v \in V_1^{\mathfrak{t},\lambda}$ or $V \in V_2^{\mathfrak{t},\lambda}$, suppose the former as the latter is analogous. Then $V_1$ contains $v$ and so is equal to $V$ since $v$ generates $V$. □

**Definition 8.74.** Let $V$ be a highest weight $\mathfrak{g}$-module. The weight of a highest weight generator of $V$ is called the **highest weight** of $V$ (by the Proposition, it is well defined, and can also be characterized as the unique weight $\lambda \in \mathfrak{t}^*$ such that $\mu \leq \lambda$ for all weights $\mu$ of $V$).

**Claim 8.75.** *Let $V$ be an irreducible (not necessarily finite-dimensional) highest weight $\mathfrak{g}$-module. Then all highest weight vectors in $V$ are highest weight generators.*

*Proof.* This claim is clear, because $V$ is irreducible any non-zero vector in $V$ is a generator of $V$. ☐

One can give here an example of a highest weight $\mathfrak{sl}_2$-module which admits two different highest weights. By the above material, it is necessarily infinite-dimensional and not irreducible (but indecomposable). It has a highest weight generator with weight 0, and also a highest weight vector (which is not a generator) with weight $-\alpha$ where $\alpha$ is the positive root (this weight is given by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto -2$). But by our convention, we speak of 0 as the highest weight of this module, and we don't call $-\alpha$ a highest weight for this module.

**Proposition 8.76.** *Let $V_1, V_2$ be two irreducible highest weight $\mathfrak{g}$-modules. Suppose that the highest weights of $V_1$ and $V_2$ coincide. Then $V_1$ and $V_2$ are isomorphic.*

*Proof.* Let us consider highest weight generators $v_1 \in V_1$ and $v_2 \in V_2$. Then $(v_1, v_2)$ is a highest weight vector in $V_1 \oplus V_2$. Let $W \subset V_1 \oplus V_2$ be the $\mathfrak{g}$-submodule generated by $(v_1, v_2)$. Then $(v_1, v_2)$ is a highest weight generator of $W$. Let us consider the projection morphism $W \hookrightarrow V \to V_1$. It is surjective, since its image contains $v_1$. It is also injective, because its kernel can be identified with a submodule of $V_2$, and so must be 0 since $V_2$ is irreducible (it can not be $V_2$ because then, for example, the morphism would be 0, and so could not be surjective). So we have an isomorphism of $W$ with $V_1$. Completely analogously, we obtain an isomorphism of $W$ with $V_2$, showing that $V_1$ and $V_2$ are isomorphic. ☐

So, now one wonders for which weights $\lambda \in \mathfrak{t}^*$, there exists an irreducible highest weight $\mathfrak{g}$-module whose highest weight is $\lambda$.

**Lemma 8.77.** *Let $\lambda \in \mathfrak{t}^*$. Suppose that there exists a highest weight $\mathfrak{g}$-module $V$ whose highest weight is $\lambda$. Then there exists an irreducible highest weight $\mathfrak{g}$-module whose highest weight is $\lambda$.*

*Proof.* Fill in... Should take quotient by all proper highest weight submodules... ☐

**Claim 8.78.** *Let $\lambda \in \mathfrak{t}^*$. Then there exists a highest weight $\mathfrak{g}$-module whose highest weight is $\lambda$. In particular, by the previous lemma, there exists therefore an irreducible highest weight $\mathfrak{g}$-module whose highest weight is $\lambda$ (unique up to isomorphism, by a previous proposition).*

Now one wonders for which weight $\lambda$, the irreducible highest weight $\mathfrak{g}$-module with highest weight $\lambda$ is finite-dimensional. Answering this, one obtains a parametrization of irreducible finite-dimensional $\mathfrak{g}$-modules.

**Claim 8.79.** *Let $\lambda \in \mathfrak{t}^*$ and let $E$ be the irreducible highest weight $\mathfrak{g}$-module with highest weight $\lambda$. Assume that $E$ is finite-dimensional. Then $\lambda(E_{i,i} - E_{i+1,i+1}) \in \mathbb{Z}_{\geq 0}$ for all $1 \leq i \leq n-1$.*

*Proof.* The matrices $E_{i,i} - E_{i+1,i+1}, E_{i,i+1}, E_{i+1,i}$ span a copy of $\mathfrak{sl}_2$, then use the knowledge of $\mathfrak{sl}_2$-modules complete... □

And the (local) culmination is:

**Proposition 8.80.** *Let $\lambda \in \mathfrak{t}^*$ and let $E$ be the irreducible highest weight $\mathfrak{g}$-module with highest weight $\lambda$. Assume that $\lambda(E_{i,i} - E_{i+1,i+1}) \in \mathbb{Z}_{\geq 0}$ for all $1 \leq i \leq n-1$. Then $E$ is finite-dimensional.*

To be continued... Time ended.