Introduction to category theory

(notes for a course taught at HUJI, Fall 2020-2021 and Fall 2022-2023)

(UNPOLISHED DRAFT)

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> It is never true that two substances are entirely alike, differing only in being two rather than one[1].
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> G. W. Leibniz, *Discourse on metaphysics*

[1]This can be imagined to be related to at least two of our themes: the imperative of considering a contractible groupoid of objects as an one single object, and also the ideology around Yoneda's lemma ("no two different things have all their properties being exactly the same").

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1 The basic language

1.1 Categories

1.1.1

In a **set**, such as the set of real numbers $\mathbb{R}$, any given two elements are either **equal** or not. Given $r_1, r_2 \in \mathbb{R}$, the answer to the question whether r_1 is equal to r_2 is either "yes" or "no", i.e. it is a truth value. Moreover, equality is an equivalence relation - (1) $r_1 = r_1$ (2) $r_1 = r_2$ and $r_2 = r_3$ imply $r_1 = r_3$ (3) $r_1 = r_2$ implies $r_2 = r_1$.

In contrast, the **class** of vector spaces over $\mathbb{R}$ is of a different nature. We don't really want to ask whether two given vector spaces V_1 and V_2 are equal or not. Instead, what is more sensible is to ask whether V_1 is **isomorphic** to V_2. The answer to that is also a "yes" or a "no", but in fact a more correct question is how V_1 is isomorphic to V_2 (informally, how V_1 is "the same as" V_2), and the answer to that question is an isomorphism between V_1 and V_2. So, the answer to the more correct question lies in a set (the set of isomorphisms between V_1 and V_2), rather than being a truth value. Of course, one can recover the truth value from the set - if the set is empty the truth value is "no", while if the set is non-empty the truth value is "yes".

The analog of equality being an equivalence relation is here - (1) There is a preferred way of making V_1 the same as V_1, the **identity** isomorphism (2) If we have a way of making V_1 the same as V_2 and a way of making V_2 the same as V_3 then we get a way of making V_1 the same as V_3, by **composition** of isomorphisms (3) If we have a way of making V_1 the same as V_2 then we get a way of making V_2 the same as V_1, taking the **inverse** of an isomorphism.

As an additional layer to the above, we also have "attempts" at pointing how vector spaces are the same, namely non-invertible linear transformations. These one can also compose, but not invert. The definition of a **category** will formalize this situation.

Some side point to make is that from our point of view, we usually don't really care at first whether a set is a set in the sense of axiomatic set theory, for example whether it has a cardinality. A set in that latter sense, of having cardinality and so on, we will call a **small set**. Rather, our main differentiating feature is that elements of a set are either equal or not, while one can not ask this question about elements in a general class (such as that of vector spaces over $\mathbb{R}$). For example, if we consider the class whose element is data (T_V) consisting of a linear endomorphsim T_V of V for every $\mathbb{R}$-vector space V, we would like to think of this class as a set, because clearly we can ask whether (T_V) and (S_V) are equal to not - this is the same as asking whether $T_V = S_V$ for all V, and this is a legitimate question since, for any given V, both T_V and S_V lie in the same set, the set of endomorphisms of V. However, the set of all data (T_V) is not a small set. Another example of a set which is not a small set is set of isomorphism classes of vector spaces over $\mathbb{R}$ - we define an equivalence relation on the class of vector spaces over $\mathbb{R}$, by saying that two vector spaces are equivalent if there exists an isomorphism between them, and we consider the class of equivalence classes, which clearly one should consider as a set (there is a notion of equality in it). However, that set has no cardinality, since it is basically the set of all cardinalities, and so introducing its cardinality will lead to paradoxes.

1.1.2

A **category** $\mathcal{C}$ consists of the following:

1. A class which might be denoted $\mathrm{Ob}(\mathcal{C})$, but we will simply, by an abuse of notation, denote it by $\mathcal{C}$ - the class of **objects**,.
2. For any two objects $X, Y \in \mathcal{C}$ a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ (or simply $\mathrm{Hom}(X, Y)$ if $\mathcal{C}$ is clear from the context) - the set of **morphisms from** X **to** Y. One also writes $X \xrightarrow{f} Y$, or $f : X \to Y$, for $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.
3. For any three objects $X, Y, Z \in \mathcal{C}$ a map of sets

$$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z),$$

   - the **composition of morphisms**. We denote by $g \circ f$ the result of applying composition to (g, f).

One requires the following:

- For $X, Y, Z, W \in \mathcal{C}$ and $f : X \to Y,\ g : Y \to Z,\ h : Z \to W$, one has $h \circ (g \circ f) = (h \circ g) \circ f$.
- For $X \in \mathcal{C}$ there exists an element $\epsilon \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ such that for all $Y \in \mathcal{C}$ one has $f \circ \epsilon = f$ for all $f : X \to Y$ and $\epsilon \circ f = f$ for all $f : Y \to X$.

It is easy to see that ϵ as above is unique; It is denoted id_X and called the **identity morphism (of** X**)**.

1.1.3

Examples of categories: The category of small sets, denoted by Set, has small sets as objects, and morphisms between a set X and a set Y are arbitrary functions $X \to Y$. Composition is the usual composition of functions. In a similar fashion, we have categories Grp/Vec_k/Top, whose objects are groups/vector spaces over a field k/topological spaces and morphisms in which are group homomorphisms/k-linear transformations/continuous maps, and much more examples of this "concrete" sort, where the objects are sets with some extra structure, and morphisms are maps of sets which respect the extra structure in the appropriate way. Here, a group means a group whose underlying set is a small set, and similarly in all the examples.

Sometimes there are various sensible choices for morphisms. For example, we can consider the category Met of metric spaces, where morphisms are continuous maps. However, we also have categories of metric spaces where morphisms are isometries (maps $f : X \to Y$ satisfying $d(f(x_1), f(x_2)) = d(x_1, x_2)$), or Lipschitz maps with constant 1 (maps $f : X \to Y$ satisfying $d(f(x_1), f(x_2)) \leq d(x_1, x_2)$), and so on. It is important to distinguish all these categories, although they have the same class of objects.

Another example is, given a partially ordered set $(P, \leq)$, the category whose class of objects is the set P, and in which the set of morphisms from $p_1 \in P$ to $p_2 \in P$ is empty if $p_1 \not\leq p_2$ and is the set $*$ with one element if $p_1 \leq p_2$. Composition of morphisms is then uniquely defined.

A simple way of obtaining a new category from an old one is passing to the **opposite category**. Namely, given a category $\mathcal{C}$, we can construct a category $\mathcal{C}^{\mathrm{op}}$ whose objects are the same as objects of $\mathcal{C}$, but where we set $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(Y, X)$ and composition is defined using that of $\mathcal{C}$ in a straight-forward manner.

Another common way of obtaining a new category from an old one is as follows. Let $\mathcal{C}$ be a category and let $\mathcal{C}^0$ be a subclass of the class of objects of $\mathcal{C}$. We then can make $\mathcal{C}^0$ a category by letting morphisms and composition to be the same as they were in $\mathcal{C}$. Thought of as a category in this way, $\mathcal{C}^0$ is said to be a **full subcategory of $\mathcal{C}$ (spanned by the class of objects $\mathcal{C}^0$)**.

1.1.4

A morphism $f : X \to Y$ is called **left invertible** (resp. **right invertible**) if there exists $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ (resp. $f \circ g = \mathrm{id}_Y$) - such a g is then called a left inverse (resp. right inverse) for f. The morphism $f : X \to Y$ is called an **isomorphism** if it is both left invertible and right invertible (one writes usually $f : X \xrightarrow{\sim} Y$ to indicate that $f : X \to Y$ is an isomorphism) . It is easy to see that if f is an isomorphism then there exists a unique left inverse for f, and a unique right inverse for f, and these are equal. This uniquely characterized morphism is denoted $f^{-1} : Y \to X$ (the **inverse**).

For example, isomorphisms in the category Set are just **bijections**. Isomorphisms in the category Top are **homeomorphisms** (so those are not continuous bijections, but rather continuous bijections whose inverse is also continuous). Isomorphisms in the category Grp are what are usually called **isomorphisms of groups**.

The ideology is that two objects, for which we have specified an isomorphism, become "the same". However, a different isomorphism between these two objects will make them "the same" in a different way, and it is important to keep track of the way in which we decide objects are the same, not just the mere fact of having a way of making them "the same".

1.1.5

A category is called a **groupoid** if every morphism in it is an isomorphism. As we explained above, one can think of the notion of a groupoid as a generalization of the notion of a set, where the answer to the question of how two elements are the same lies in a set, instead of being a truth value.

To every category $\mathcal{C}$ one can naturally associate a groupoid $\mathcal{C}^{\mathrm{grpd}}$ (the **core groupoid**) by keeping the same objects and keeping all isomorphisms, discarding morphisms which are not isomorphisms.

1.1.6

Given a set S, we construct a groupoid whose class of objects is the set S, and the set of morphisms from $s_1 \in S$ to $s_2 \in S$ is empty if $s_1 \neq s_2$ and is the set $*$ with one element if $s_1 = s_2$. Composition of morphisms is uniquely defined (formally, this example is subsumed by the above example of attaching a category to a partially ordered set).

Another basic example of a groupoid is as follows. Let G be a group. We construct a groupoid $G\backslash *$, whose class of objects is the set $*$ with one element, and with morphisms $\mathrm{Hom}_{G\backslash *}(*, *) := G$. Composition is by multiplication in the group.

1.1.7

A morphism $f : X \to Y$ is called a **monomorphism** (resp. **epimorphism**) if for every $Z \in \mathcal{C}$ the map $\mathrm{Hom}_{\mathcal{C}}(Z, X) \to \mathrm{Hom}_{\mathcal{C}}(Z, Y)$ given by $g \mapsto f \circ g$ (resp. the map $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ given by $g \mapsto g \circ f$) is injective.

Exercise 1.1. *A left invertible morphism is a monomorphism (resp. a right invertible morphism is an epimorphism). Thus, an isomorphism is both a monomorphism and an epimorphism.*

An example of a morphism which is both a monomorphism and an epimorphism, but not an isomorphism, is a morphism of topological spaces, which is injective and whose image is dense in the target (but which is not a homeomorphism).

1.1.8

Let $\mathcal{C}$ be a category. One can formalize the process of answering the question of whether two objects are isomorphic, which is cruder than the question of how the two objects are isomorphic, as follows. Isomorphism of objects in $\mathcal{C}$ is an equivalence relation, and the set of equivalence classes we will denote by $\pi_0(\mathcal{C})$ (this is the set of **isomorphism classes in** $\mathcal{C}$). Notice that this is indeed a set, not a class. Thus, $\mathcal{C} \mapsto \pi_0(\mathcal{C})$ is the standard process of "crudifying" a category into a set.

1.1.9

A common way of obtaining a full subcategory of a category $\mathcal{C}$ is by considering a subset Δ of the set $\pi_0(\mathcal{C})$ of isomorphism classes in $\mathcal{C}$, and considering the full subcategory $\mathcal{C}^0 \subset \mathcal{C}$ whose objects are those whose isomorphism class lies in Δ.

1.2 Functors

1.2.1

Let $\mathcal{C}, \mathcal{D}$ be categories. A **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of the following:

1. For any $X \in \mathcal{C}$ an $F(X) \in \mathcal{D}$.
2. For any $X, Y \in \mathcal{C}$, a map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$. The image of $X \xrightarrow{f} Y$ under this map we denote by $F(f)$.

One requires the following:

- For $X, Y, Z \in \mathcal{C}$ and $f : X \to Y, g : Y \to Z$ one has $F(g \circ f) = F(g) \circ F(f)$.
- For $X \in \mathcal{C}$ one has $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

1.2.2

One class of examples of functors are so-called **forgetful functors**. For example, we have the functor Grp $\to$ Set sending a group G to G considered as a set (i.e. with the group structure, the multiplication, "forgotten"), and a homomorphism of groups it sends to itself considered as a map of sets. One has a lot of similar examples.

So, given a field k, we have the forgetful functor $\mathrm{Vec}_k \to \mathrm{Set}$. Does one have some sensible functor in the other direction, a functor $\mathrm{Set} \to \mathrm{Vec}_k$? Given a set S, we can consider the formally fabricated vector space $k[S]$ with basis S (so it consists of formal combinations $\sum_{s \in S} c_s \cdot \delta_s$, where δ_s is a formal symbol ("delta function" or, slightly better, "delta measure") created for every $s \in S$, $c_s \in k$ are scalars, and all the c_s's except finitely many are equal to 0). Given a map of sets $f : S \to T$, we have a natural k-linear transformation $k[S] \to k[T]$ (sending δ_s to $\delta_{f(s)}$). This defines a functor $\mathrm{Set} \to \mathrm{Vec}_k$. In some sense, this functor makes out of a set a k-vector space "in the most efficient way". A formalization of that will come later (stating that this functor $S \mapsto k[S]$ is left adjoint to the forgetful functor $\mathrm{Vec}_k \to \mathrm{Set}$).

One of the first conscious appearances of functors in mathematics (as far as we understand) was as constructions of "algebraic invariants" of topological spaces. For example, denoting by AbGrp the full subcategory of Grp consisting of abelian groups, there is a functor $H_1 : \mathrm{Top} \to \mathrm{AbGrp}$ (the **first homology with integral coefficients**). This functor, roughly speaking, records "two-dimensional holes" in a topological space. For example, the two-dimensional unit disc $D_2 \subset \mathbb{R}^2$ has no holes, so $H_1(D_2) = 0$. The unit circle $S^1 \subset \mathbb{R}^2$ has one hole, so $H_1(S^1) \cong \mathbb{Z}$. A nice illustration of the power of this is proving **Brouwer's fixed point theorem**. It (or a special case of it) says that every continuous map $f : D_2 \to D_2$ admits a fixed point - there exists $x \in D_2$ such that $f(x) = x$. The proof is by contradiction. Assuming the contrary, we can define a continuous map $g : D_2 \to S^1$ by sending every $x \in D_2$ to the point on S^1 to which one eventually arrives if one goes, on the line passing through x and $f(x)$, from $f(x)$ in the direction of x. Notice that if $x \in S^1$, then $g(x) = x$. In other words, we have in Top morphisms

$$S^1 \xrightarrow{i} D_2 \xrightarrow{g} S^1,$$

where i is the inclusion, and the composition $g \circ i$ is equal to id_{S^1}. We now apply the functor H_1, and obtain in AbGrp morphisms

$$H_1(S^1) \xrightarrow{H_1(i)} H_1(D_2) \xrightarrow{H_1(g)} H_1(S^1)$$

whose composition is $H_1(\mathrm{id}_{S^1}) = \mathrm{id}_{H_1(S^1)}$, or according to what we said before morphisms

$$\mathbb{Z} \to 0 \to \mathbb{Z}$$

whose composition is $\mathrm{id}_{\mathbb{Z}}$. However, the composition of those is the zero morphism $0_{\mathbb{Z}}$, and $0_{\mathbb{Z}} \neq \mathrm{id}_{\mathbb{Z}}$, so that we arrive to a contradiction.

1.2.3

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Notice, as a small exercise, that if two objects in $\mathcal{C}$ are isomorphic, then so are their images in $\mathcal{D}$ under F. Thus F induces naturally a map $\pi_0(\mathcal{C}) \to \pi_0(\mathcal{D})$. We refer to the image of this map, or to the corresponding full subcategory of $\mathcal{D}$, as the **essential image** of F. Thus, concretely, an object in $\mathcal{D}$ belongs to the essential image of F if it is isomorphic to some object of the form $F(X)$, for $X \in \mathcal{C}$. One says that F is **essentially surjective** if its essential image is the whole of $\mathcal{D}$.

For example, the functor Set $\to \mathrm{Vec}_k$ sending S to $k[S]$, that we considered before, is essentially surjective; this is equivalent to a basic theorem in linear algebra, that every vector space has a basis! Here one can reiterate, that this means that every $V \in \mathrm{Vec}_k$ is isomorphic to $k[S]$ for some set S, but it is meaningless from our perspective to ask whether every $V \in \mathrm{Vec}_k$ is equal to $k[S]$ for some S (thus, from our perspective, we don't want to be able to ask this question and the answer being negative; we want ideally to not be able even to ask this question[2]).

[2]Informally, some things are worse than being wrong, they are "not even wrong" (what is the source of this expression? I remeber vaguely that maybe someone called string theory like that, but I am not sure).

1.2.4

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. One says that F is **faithful / full / fully faithful** if for every $X, Y \in \mathcal{C}$, the map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is injective / surjective / bijective. Sometimes one might use the term "embedding" for a fully faithful functor.

An example of a faithful but not full functor is some typical forgetful functor, such as the forgetful functor Grp $\to$ Set. Another example is the category of metric spaces with isometries sitting in the category Met of metric spaces with continuous maps.

Examples of full but not faithful functors can by typically obtained by putting some equivalence relation on morphisms. Let us be given a category $\mathcal{C}$ and for every $c_1, c_2 \in \mathcal{C}$ an equivalence relation $\sim$ on $\mathrm{Hom}_{\mathcal{C}}(c_1, c_2)$, such that $f_1 \sim f_2$ implies $f_1 \circ g \sim f_2 \circ g$ (where $g :? \to c_1$) and $g \circ f_1 \sim g \circ f_2$ (where $g : c_2 \to ?$). Then we can define a category $\mathcal{C}/\sim$, whose objects are the objects of $\mathcal{C}$, and $\mathrm{Hom}_{\mathcal{C}/\sim}(c_1, c_2) := \mathrm{Hom}_{\mathcal{C}}(c_1, c_2)/\sim$. Composition is given by choosing representatives and then taking the equivalence class of their composition (and this is well-defined thanks to the condition above). Then we have a natural functor $\mathcal{C} \to \mathcal{C}/\sim$ (which is identity on objects), which is full, but not in general faithful (unless all the equivalence relations are trivial). Here are examples of such constructs. We can take $\mathcal{C} := \mathrm{Vec}_k$ and say that $T \sim S$ if $\dim \mathrm{Im}(T - S) < \infty$ (working in the quotient category forces us to ignore finite-dimensional information, so to speak). We can take $\mathcal{C} :=$ Top, and say, given $f_1, f_2 : X_1 \to X_2$, that $f_1 \sim f_2$ if f_1 is **homotopic** to f_2. Recall, this means that there exists a continuous $F : [0, 1] \times X_1 \to X_2$ such that $F(0, -) = f_1(-)$ and $F(1, -) = f_2(-)$. The resulting quotient category is the **homotopy category** HoTop.

An example of a fully faithful functor is the inclusion functor of a full subcategory in a category. Another example is the functor Met $\to$ Top (sending a metric space to itself viewed as a topological space).

1.2.5

We can compose functors: If we have functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, we can naturally form a functor $G \circ F : \mathcal{C} \to \mathcal{E}$. This has various straight-forward properties that we omit.

1.3 Natural transformations

1.3.1

Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A **natural transformation** (or simply a **morphism (of functors)**) $\alpha : F \to G$ consists of a morphism $\alpha_X : F(X) \to G(X)$ for every $X \in \mathcal{C}$, such that:

- For $f : X \to Y$ in $\mathcal{C}$ one has $G(f) \circ \alpha_X = \alpha_Y \circ F(f)$. One expresses this condition usually by saying that the following is a commutative square:

$$\begin{array}{ccc} F(X) & \xrightarrow{\alpha_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\alpha_Y} & G(Y) \end{array} .$$

Another very common terminology is as follows. When we are given morphisms $\alpha_X : F(X) \to G(X)$ for all $X \in \mathcal{C}$, we say that these are **functorial in** X if these form a natural transformation, i.e. the condition of commutativity above is satisfied.

1.3.2

Given $F, G, H : \mathcal{C} \to \mathcal{D}$, if we have natural transformations $\alpha : F \to G$ and $\beta : G \to H$ it is clear how to define a natural transformation $\beta \circ \alpha : F \to H$. In this way, we can form the **category of functors** $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ between $\mathcal{C}$ and $\mathcal{D}$. Its objects are functors $\mathcal{C} \to \mathcal{D}$. For $F, G : \mathcal{C} \to \mathcal{D}$, the set of morphisms $\mathrm{Hom}(F, G)$ is the set of natural transformations $F \to G$ (notice that this is indeed a set in our sense, although it can easily be non-small in this generality!). As we said, it is straight-forward how to define composition, and to check that there exist identities.

Exercise 1.2. *Let $F, G \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Let $\alpha : F \to G$ be a morphism. Show that α is an isomorphism if and only if $\alpha_X : F(X) \to G(X)$ is an isomorphism (in $\mathcal{D}$) for all $X \in \mathcal{C}$.*

1.3.3

As an example of a category of functors, we can consider the category $\mathcal{A}$ whose set of objects is $\{*_1, *_2\}$ and the only non-identity morphism is one morhpism $*_1 \to *_2$. We consider then the category $\mathrm{Fun}(\mathcal{A}, \mathcal{C})$. Its objects are morphisms in $\mathcal{C}$, i.e. triples (c_1, c_2, α) consisting of $c_1, c_2 \in \mathcal{C}$ and $\alpha : c_1 \to c_2$. A morphism in this category, from (c_1, c_2, α) to (c_1', c_2', α') is a pair of morphisms (β, γ) fitting in a **commutative diagram**

$$\begin{array}{ccc} c_1 & \xrightarrow{\alpha} & c_2 \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \gamma} \\ c_1' & \xrightarrow{\alpha'} & c_2' \end{array} .$$

For example, given a field k, denoting by Vec_k^{fd} the full subcategory of Vec_k consisting of vector spaces of finite-dimension, think what is $\pi_0(\mathrm{Fun}(\mathcal{A}, \mathrm{Vec}_k^{fd}))$. Elements of this set are encoded by three numbers - the dimension of the source, the dimension of the target and the rank of the transformation between them.

This is the "first part of linear algebra". One can also consider the category $\mathcal{B}$ with one object $*$ and $\mathrm{Hom}_{\mathcal{B}}(*,*) := \mathbb{Z}_{\geq 0}$ with composition being addition. Then we can identify objects of $\mathrm{Fun}(\mathcal{B},\mathcal{C})$ with pairs (c,α) consisting of $c \in \mathcal{C}$ and $\alpha : c \to c$ (this α is where $1 \in \mathrm{Hom}_{\mathcal{B}}(*,*)$ goes; Then n goes to α^n, so this information is redundant). A morphism from (c,α) to (c',α') is a morphism β fitting in a commutative diagram

$$\begin{array}{ccc} c & \xrightarrow{\alpha} & c \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta} \\ c' & \xrightarrow{\alpha'} & c' \end{array} .$$

Now thinking about $\pi_0(\mathrm{Fun}(\mathcal{B},\mathrm{Vec}_k^{fd}))$ is already the "second part of linear algebra" - concretely, the theory of square matrices up to similarity. To describe this we already need Jordan form etc.

1.3.4

Let us give another example of a category of functors. Let G be a group and consider the groupoid $G\backslash *$ as before. Let $\mathcal{C}$ be a category. Then $\mathcal{C}^G := \mathrm{Fun}(G\backslash *,\mathcal{C})$ is the category of objects in $\mathcal{C}$ equipped with an action of G. Namely, we can explicitly describe an object in $\mathcal{C}^G$ as the data $(c,(\alpha_g)_{g\in G})$ where $c \in \mathcal{C}$ and $\alpha_g : c \xrightarrow{\sim} c$, such that $\alpha_1 = \mathrm{id}_c$ and $\alpha_{g_1g_2} = \alpha_{g_1} \circ \alpha_{g_2}$. A morphism from such $(c,(\alpha_g)_{g\in G})$ to another $(c',(\alpha'_g)_{g\in G})$ is a morphism $\beta : c \to c'$ in $\mathcal{C}$ which satisfies $\beta \circ \alpha_g = \alpha'_g \circ \beta$ for all $g \in G$. For example, if $\mathcal{C} := \mathrm{Set}$, we obtain the category G-Set $:= \mathrm{Set}^G$ of **G-sets**. If $\mathcal{C} := \mathrm{Vec}_k$, we obtain the category $\mathrm{Rep}_k(G) := \mathrm{Vec}_k^G$ of **representations of the group G over the field k**.

1.3.5

Composition of functors and natural transformations have various straightforward interaction properties, which we omit.

2 Equivalence of categories

2.1 Contractible groupoids

A set S with one element is characterized as follows: S is non-empty (i.e. there exists an element in S), and in addition if $x, y \in S$ then $x = y$. Let us call such a set **contractible**.

There is an analogous notion for groupoids: Let $\mathcal{S}$ be a groupoid. We say that it is **contractible** if there exists an object in $\mathcal{S}$, and for every two objects $x, y \in \mathcal{S}$ the set $\mathrm{Hom}_{\mathcal{S}}(x,y)$ is contractible (i.e. there exists a unique isomorphism from x to y). We say that a category is **contractible** if it is a groupoid and it is contractible (equivalently, if the class of objects is non-empty, and between any two objects there exists a unique morphism).

A philosophy of utmost importance is that a contractible groupoid can be thought of as the trivial groupoid $*$ (admitting one object and one morphism). One should think that there is precisely one way of specifying an object in a contractible groupoid (you pick any object; If someone picked another object, there is a unique identification, isomorphism, between these two objects, so we are OK). How this philosophy fits into the current formal framework of axiomatic mathematics is less satisfactory (we want to say that there is no choice when wanting to specify an object in a contractible groupoid, however current axiomatics require us, in order to have at our hands a determined object of this groupoid, to choose one). As a side remark, Voevodsky's homotopy type theory tries to provide an axiomatic system in which this philosophy is respected.

Example 2.1. *So, the category with two objects $*_1$ and $*_2$, where the morphisms are the identity morphism of $*_1$, the identity morphism of $*_2$, one morphism from $*_1$ to $*_2$ and one morphism from $*_2$ to $*_1$, is contractible. One should think that it has one object, because any two objects in it are canonically identified (there exists a unique isomorphism between them). However, in the standard axiomatic framework, to work with that one object one has to name it, say x, and then who, from the elements of the class $\{*_1, *_2\}$ of objects of our category is x - $*_1$ or $*_2$? This is a matter of the **language not being adapted** to what we desire to express.*

Example 2.2. *One can consider the full subcategory of* Set *consisting of sets with one element. It is contractible ("all sets with one element are the same"). Similarly, the full subcategory of* Set *consisting of empty sets is contractible. However, the full subcategory of* Set *consisting of sets which have two elements is not contractible.*

Example 2.3. *Let G be a non-trivial group, for example let us take it to be the group with two elements. Then the groupoid $G\backslash *$ considered above is not contractible. It has one element, but this element can be identified with itself in several ways (two ways in our example).*

Remark 2.4. A standard phrasing, given a contractible groupoid, is to say that an object in it is **unique, up to a unique isomorphism**.

2.2 Fibers

Recall that given a map of sets $f : S \to T$ and $t \in T$, the **fiber** $f^{-1}(t)$ is the set $\{s \in S \mid f(s) = t\}$. What is the analog for categories? The definition will illustrate well the **categorical thinking** required when operating with categories.

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Let $d \in \mathcal{D}$. We want to define a category $F^{-1}(d)$, the **fiber** of F over d. What should be an object of it? It is not correct for it to be an object $c \in \mathcal{C}$ such that $F(c)$ is isomorphic to d, because in categorical thinking we should try always remember how objects are identified, not only whether they are identifiable. Thus, an object of $F^{-1}(d)$ is a pair

(c, α) consisting of an object $c \in \mathcal{C}$ and an isomorphism $F(c) \xrightarrow{\alpha} d$. A morphism between (c, α) and (c', α') is a morphism $\beta : c \to c'$ in $\mathcal{C}$ such that $\alpha' \circ F(\beta) = \alpha$, i.e. the following diagram commutes:

$$\begin{array}{ccc} F(c) & \xrightarrow{F(\beta)} & F(c') \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\alpha'} & \\ & d & \end{array} .$$

I have now a bit of a pedagogical problem of what examples of fibers to give, what extra terminology to introduce, how much "sophistication" to insert at this particular point, etc. So the following examples and remarks perhaps should be modified at some point. Anyway, they are not strictly neccesary at this point, so a reader might skip them.

Example 2.5. *A first approximation to understanding a category $\mathcal{C}$ is the understanding of the set $\pi_0(\mathcal{C})$, so let us understand those for some fibers. Let us consider the forgetful functor $F : \mathrm{Grp} \to \mathrm{Set}$. Check that $\pi_0(F^{-1}(X))$ is in a canonical bijection with the set of **group structures** on X (i.e. maps $X \times X \to X$ satisfying the group axioms). Similarly, for the forgetful functor $F : \mathrm{Top} \to \mathrm{Set}$, check that $\pi_0(F^{-1}(X))$ is in a canonical bijection with the set of **topologies** on X. Next, let us consider the example of $F : \mathrm{Set} \to \mathrm{Vec}_k$ given by $F := k[-]$ which we considered before. We can check that $\pi_0(F^{-1}(V))$ is in a canonical bijection with the set of **bases** of V.*

Remark 2.6. Let us say that a category $\mathcal{C}$ is **set-like** if it is a groupoid and for every $X, Y \in \mathcal{C}$ we have $|\mathrm{Hom}_{\mathcal{C}}(X, Y)| \leq 1$. Thus, a contractible category is set-like. Also, given a set, we associated to it a groupoid in a natural way in §1.1.6; it is set-like. We will explain in Remark 2.17 that if a category $\mathcal{C}$ is set-like, we can think of it as carrying precisely the same information as the set $\pi_0(\mathcal{C})$ (to justify this formally we need to have the notion of equivalence of categories which we will discuss soon). A functor $F : \mathcal{C} \to \mathcal{D}$ is called **conservative** if given any morphism $\alpha : X \to Y$ in $\mathcal{C}$, if $F(\alpha)$ is an isomorphism then α is an isomorphism. One can do a simple exercise, that the fibers of a conservative functor are set-like, and therefore to understand them is the same as to understand their $\pi_0(-)$'s. The forgetful functor $\mathrm{Grp} \to \mathrm{Set}$ and the functor $k[-] : \mathrm{Set} \to \mathrm{Vec}_k$ are conservative.

Remark 2.7. The forgetful functor $\mathrm{Top} \to \mathrm{Set}$ is not conservative. And indeed its fibers are generally not set-like. What are they then? Let us say that a category $\mathcal{C}$ is **partially-ordered-set-like** if for every $X, Y \in \mathcal{C}$ we have $|\mathrm{Hom}_{\mathcal{C}}(X, Y)| \leq 1$. Thus, a set-like category is a partially-ordered-set-like category which is in addition a groupoid. Understand how, given a partially-ordered-set-like category $\mathcal{C}$, to construct canonically a partial order on $\pi_0(\mathcal{C})$. Again, when we will have the notion of an equivalence of categories, we will be able to explain in which sense the information carried in a partially-ordered-set-like category $\mathcal{C}$ is precisely the information carried in $\pi_0(\mathcal{C})$ together with the partial order it carries. One immediately checks that the fibers of a faithful functor are partially-ordered-set-like. In particular, the forgetful functor $F : \text{Top} \to \text{Set}$ is faithful and so its fibers are partially-ordered-set-like. Understand that $\pi_0(F^{-1}(X))$, viewed as a partially ordered set, is in a canonical isomorphism of partially ordered sets with the partially ordered set of topologies on X, where a topology is $\leq$ another topology if it is finer than it.

Example 2.8. *Here is another example of a fiber. Let G be a group. Let us denote by $\bullet$ the category with one object and the identity morphism from it to itself, and nothing else. By some abuse of notation, we denote by $\bullet$ the object in $\bullet$, and by $*$ the object in $G\backslash *$. Consider the unique functor $F : \bullet \to G\backslash *$. What is $F^{-1}(*)$? Unfolding definitions, we see that the class of objects of $F^{-1}(*)$ is in a natural bijection with the set G, and the morphisms we have in $F^{-1}(*)$ are solely the identity morphisms. So $F^{-1}(*)$ can be thought of as the groupoid associate with the set G. One thinks of this, informally, as "the point divided by G-symmetry has a cover by the point, and the fiber is G". Thus, informally, "the map F is $|G|$-to-1". Thus, it makes some sense, perhaps, to think of $G\backslash *$ as "having $1/|G|$ elements".*

2.3 Fibers and fully faithfulness

Of course, a map of sets $f : S \to T$ is injective if and only if all its fibers are either empty or with one element. We now discuss the analog for categories.

Lemma 2.9. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If F is fully faithful, then all fibers of F are either empty or contractible.*

Proof. Let $d \in \mathcal{D}$. Let $(c, \alpha), (c', \alpha') \in F^{-1}(d)$. We need to show that there is exactly one morphism from (c, α) to (c', α'). By definition, such morphisms are morphisms $\beta : c \to c'$ such that $\alpha' \circ F(\beta) = \alpha$, i.e. $F(\beta) = (\alpha')^{-1} \circ \alpha$. Since F is fully faithful, the map $\text{Hom}_{\mathcal{C}}(c, c') \to \text{Hom}_{\mathcal{D}}(F(c), F(c'))$ induced by F, i.e. $\beta \mapsto F(\beta)$, is bijective. Hence the existence and uniqueness of β as required follows. ☐

Remark 2.10. Of course, a fiber $F^{-1}(d)$ is non-empty precisely when d lies in the essential image of F (by definition).

For groupoids, which are more direct generalizations of sets, we also have the converse:

Lemma 2.11. *Suppose that $F : \mathcal{C} \to \mathcal{D}$ is a functor, with $\mathcal{D}$ a groupoid. If all fibers of F are either empty or contractible, then F is fully faithful.*

Proof. Let $c_1, c_2 \in \mathcal{C}$. Given $\alpha : F(c_1) \to F(c_2)$, we want to show that there exists a unique $\beta : c_1 \to c_2$ such that $F(\beta) = \alpha$. In the fiber $F^{-1}(F(c_2))$, we have two objects - $(c_2, \text{id}_{F(c_2)})$ and (c_1, α) (of course, α is an isomorphism since $\mathcal{D}$ is a groupoid). By the given, we have a unique morphism from (c_1, α) to $(c_2, \text{id}_{F(c_2)})$ in $F^{-1}(F(c_2))$; unfolding the definition of the fiber, this means that there exists a unique morphism $\beta : c_1 \to c_2$ such that $\text{id}_{F(c_2)} \circ F(\beta) = \alpha$, i.e. $F(\beta) = \alpha$. ☐

2.4 A lemma on fully faithfulness in families

Oftentimes, one thinks of a functor $\mathcal{D} \to \mathcal{C}$ as a **family of objects in** $\mathcal{C}$ **parametrized by** $\mathcal{D}$. One expects that if a category $\mathcal{C}^0$ embeds in a category $\mathcal{C}$, then the category of $\mathcal{D}$-families of objects in $\mathcal{C}^0$ embeds into the category of $\mathcal{D}$-families of objects in $\mathcal{C}$. We formalize this in the lemma that follows. We would like to draw attention how the proof of the lemma barely survives under the current axiomatic regime. An additional remark is that it would be perhaps ideologically pleasant to discuss the Grothendieck construction so that we can think of the lemma as expressing the idea that given a family of contractible categories, a choice of an object in each one of them, "coherent"/functorial in the family, is also a contractible choice. But we don't want to digress too much.

Lemma 2.12. *Let $I : \mathcal{C}^0 \to \mathcal{C}$ be a fully faithful functor and let $\mathcal{D}$ be a category. Then the functor*

$$\mathrm{Fun}(\mathcal{D}, \mathcal{C}^0) \xrightarrow{I\circ -} \mathrm{Fun}(\mathcal{D}, \mathcal{C})$$

is fully faithful, and its essential image consists of functors $F : \mathcal{D} \to \mathcal{C}$ such that $F(d)$ is in the essential image of I for all $d \in \mathcal{D}$.

Proof. We leave to the reader to check that our functor is indeed fully faithful (this is straight-forward). Clearly, for a functor $F : \mathcal{D} \to \mathcal{C}^0$, the functor $I \circ F$ satisfies the condition claimed to characterize objects in the image of $I \circ -$. Conversely, let $G : \mathcal{D} \to \mathcal{C}$ be such that $G(d)$ lies in the essential image of I for every $d \in \mathcal{D}$. Then, given $d \in \mathcal{D}$, since $G(d)$ lies in the essential image of I we have that $I^{-1}(G(d))$ is non-empty, and since I is fully faithful $I^{-1}(G(d))$ therefore is in fact contractible. Let us fix an object $(\widetilde{F}(d), \epsilon_d) \in I^{-1}(G(d))$ (so $\widetilde{F}(d) \in \mathcal{C}^0$ and $\epsilon_d : I(\widetilde{F}(d)) \xrightarrow{\sim} G(d)$). Here we remark that in the current axiomatics of mathematics, this is some "crazy" choice, as it is a choice for every $d \in \mathcal{D}$, requiring perhaps an axiom of choice; However, as we explained above, since $I^{-1}(G(d))$ is contractible, in fact we perceive no choice, and the issue is that our language is not suitable for what we want to express. Given a morphism $\alpha : d_1 \to d_2$, there is a unique morphism $\alpha' : \widetilde{F}(d_1) \to \widetilde{F}(d_2)$ making the following square commute, because the vertical morphisms are isomorphisms and I is fully faithful:

$$\begin{array}{ccc} G(d_1) & \xrightarrow{G(\alpha)} & G(d_2) \\ \uparrow{\scriptstyle \epsilon_{d_1}} & & \uparrow{\scriptstyle \epsilon_{d_2}} \\ I(\widetilde{F}(d_1)) & \xrightarrow{I(\alpha')} & I(\widetilde{F}(d_2)) \end{array} .$$

We denote this α' by $\widetilde{F}(\alpha)$. We leave to the reader the straight-forward check that we obtain in this way a functor $\widetilde{F} : \mathcal{D} \to \mathcal{C}^0$ (having defined it on objects and morphisms), and an isomorphism of functors $\epsilon : I \circ \widetilde{F} \xrightarrow{\sim} G$. □

2.5 Definition of equivalence of categories

In a set, we have equality. In a category, we have isomorphism. In the totality of categories, what is the correct notion of "being the same"? In other words, when does a functor between categories "identify" them? The technical terms for such a functor will be an **equivalence of categories**.

Definition 2.13. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is called an **equivalence of categories** if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ and isomorphisms of functors $\epsilon : F \circ G \xrightarrow{\sim} \mathrm{Id}_{\mathcal{D}}$ and $\delta : \mathrm{Id}_{\mathcal{C}} \xrightarrow{\sim} G \circ F$.

This definition is in the spirit "an indentification is something that has a two-sided inverse". One can also characterize equivalence of categories more in the spirit of "an identification is something that is injective and surjective":

Claim 2.14. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. The following are equivalent:*

1. *The functor F is an equivalence of categories.*
2. *The functor F is fully faithful and essentially surjective.*

Proof. (1) $\implies$ (2): There exist a functor $G : \mathcal{D} \to \mathcal{C}$ and isomorphisms of functors $\epsilon : F \circ G \xrightarrow{\sim} \mathrm{Id}_{\mathcal{D}}$ and $\delta : \mathrm{Id}_{\mathcal{C}} \xrightarrow{\sim} G \circ F$.

First, F is essentially surjective, because given any $X \in \mathcal{D}$ we have an isomorphism $\epsilon_X : F(G(X)) \to X$.

Let us next see that F is faithful. Notice that, since $G \circ F$ is isomorphic to $\mathrm{Id}_{\mathcal{C}}$ and $\mathrm{Id}_{\mathcal{C}}$ is faithful, $G \circ F$ is faithful. Hence F is faithful, by the following immediate exercise:

Exercise 2.1. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors. If $G \circ F$ is faithful then F is faithful.*

Finally, let us see that F is full. Notice that by the just established faithfulness, but applied to G instead of F, we see that G is faithful. Notice in addition that, since $G \circ F$ is isomorphic to $\mathrm{Id}_{\mathcal{C}}$ and $\mathrm{Id}_{\mathcal{C}}$ is full, $G \circ F$ is full. Hence F is full, by the following immediate exercise:

Exercise 2.2. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors. If G if faithful and $G \circ F$ is full then F is full.*

(2) $\implies$ (1): We will use Lemma 2.12 with $\mathcal{C}^0 := \mathcal{C}, \mathcal{C} := \mathcal{D}, \mathcal{D} := \mathcal{D}, I := F$ (sorry for the confusing overlap - the notations on the left are of the Lemma, while the notations on the right are the current notations). Since F is essentially surjective, the functor $\mathrm{Id}_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ is such that every object in its image lies in the essential image of F. Therefore by Lemma 2.12 there exists a functor $G : \mathcal{D} \to \mathcal{C}$ together with an isomorphism $\epsilon : F \circ G \xrightarrow{\sim} \mathrm{Id}_{\mathcal{D}}$.

It remains to show that $G \circ F$ is isomorphic to $\mathrm{Id}_{\mathcal{C}}$. Let us again use Lemma 2.12, this time with $\mathcal{C}^0 := \mathcal{C}, \mathcal{C} := \mathcal{D}, \mathcal{D} := \mathcal{C}, I := F$. In Lemma 2.12 we consider the functor

$$\mathrm{Fun}(\mathcal{C}, \mathcal{C}) \xrightarrow{F\circ -} \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \tag{2.1}$$

and state that it is fully faithful. One has the following simple exercise:

Exercise 2.3. *Let $F : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor. Let $X, Y \in \mathcal{C}$. Given an isomorphism of $F(X)$ and $F(Y)$, we can construct an isomorphism of X and Y. In particular, if $F(X)$ and $F(Y)$ are isomorphic then X and Y are isomorphic.*

This exercise, together with the fully faithfullness of the functor in (2.1), shows that in order to see that $G \circ F$ and $\mathrm{Id}_{\mathcal{C}}$, two objects in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$, are isomorphic, it is enough to see that $F \circ (G \circ F)$ and $F \circ \mathrm{Id}_{\mathcal{C}}$, two objects in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, are isomorphic. However, the latter functor is equal to F, while the former functor is isomorphic to F:

$$F \circ (G \circ F) = (F \circ G) \circ F \xrightarrow{\epsilon \overline{\circ} \mathrm{id}_F} \mathrm{Id}_{\mathcal{C}} \circ F = F.$$

Here the notation $\overline{\circ}$ is as follows: Given functors $F_1, F_2 : \mathcal{C} \to \mathcal{D}$, functors $G_1, G_2 : \mathcal{D} \to \mathcal{E}$ and morphisms of functors $\alpha : F_1 \to F_2$ and $\beta : G_1 \to G_2$, we define a morphism of functors $\beta \overline{\circ} \alpha : G_1 \circ F_1 \to G_2 \circ F_2$ in a straightforward manner (see that you understand how to define it). It has the property that if both α and β are isomorphisms, then so is $\beta \overline{\circ} \alpha$.

□

There are also some other characterizations of equivalences of categories, but we will omit a discussion of these for now.

The ideology is that all "sensible" operations with categories should not care when replacing a category by an equivalent one (as we try to emphasize, this ideology is at tension with the currently ruling axiomatics). For example:

Lemma 2.15. *Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of categories. Let $\mathcal{E}$ be a category. Then the functor*

$$\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \xrightarrow{- \circ F} \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

is an equivalence of categories.

Proof. Let us leave this as an exercise. □

What should be called the **inverse** of an equivalence of categories (sometimes called **quasi-inverse**, but we will just say inverse)? In category theory, it is important to be sensitive to differences between answers like "G such that there exists an isomorphism ..." and "G equipped with an isomorphism ...". Namely, we want the inverse to be unique up to a unique isomorphism, in other words the "pool" of inverses should form a contractible groupoid. Guided by this, one can phrase the definition of the inverse to an equivalence in several slightly different but equivalent ways, but let us choose one. Namely, given an equivalence of categories $F : \mathcal{C} \to \mathcal{D}$, let us mean by an inverse to F a functor $G : \mathcal{D} \to \mathcal{C}$ equipped with an isomorphism $\delta : \mathrm{Id}_{\mathcal{C}} \xrightarrow{\sim} G \circ F$. It is convenient to think about the category of such as the fiber over $\mathrm{Id}_{\mathcal{C}}$ of the functor

$$\mathrm{Fun}(\mathcal{D}, \mathcal{C}) \xrightarrow{- \circ F} \mathrm{Fun}(\mathcal{C}, \mathcal{C}).$$

Since by the lemma above this functor is itself an equivalence of categories, this fiber is contractible, so that indeed we obtain that an inverse to an equivalence of categories is unique up to a unique isomorphism.

Exercise 2.4. *Define the category of inverses to F in a wrong way, for example as the full subcategory of* $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ *consisting of functors $G : \mathcal{D} \to \mathcal{C}$ for which there exists an isomorphism between $\mathrm{Id}_{\mathcal{C}}$ and $G \circ F$, and show that it is then not in general contractible.*

Remark 2.16. From the above material we see that a functor $F : \mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is a groupoid, is an equivalence of categories if and only if all its fibers are contractible. This is a very clear and pleasant characterization, which we thus have when we talk about groupoids, rather than more general categories (let us remind that groupoids are the direct generalization of sets, rather than more general categories, which are perhaps more like the generalization of partially ordered sets).

Remark 2.17. We can now justify the definition of a set-like category given in Remark 2.17. Namely, recall that in §1.1.6, given a set S, we defined a groupoid, let us denote it by S^{cat}. Then one can check that a category $\mathcal{C}$ is equivalent to S^{cat} for some set S if and only if $\mathcal{C}$ is set-like. Another thing one can check is that given a groupoid $\mathcal{C}$, we have a natural functor $F : \mathcal{C} \to \pi_0(\mathcal{C})^{\mathrm{cat}}$ (see that you understand how it is defined). Then the groupoid $\mathcal{C}$ is set-like if and only if this functor is an equivalence of categories. In other words, if a category $\mathcal{C}$ is set-like, then there is a canonical equivalence of categories between $\mathcal{C}$ and $\pi_0(\mathcal{C})^{\mathrm{cat}}$. Similarly, we can justify the definition of a partially-ordered-set-like category given in Remark 2.7. Namely, recall that in §1.1.3, given a partially ordered set P, we defined a category, let us denote it by P^{cat}. One can check that a category $\mathcal{C}$ is equivalent to P^{cat} for some partially ordered set P if and only if $\mathcal{C}$ is partially-ordered-set-like.

2.6 Simple examples of equivalence of categories

2.6.1

Let $\bullet$ denote the "trivial" category, which has one object, the identity morphism from it to itself, and nothing else. We can see that a category $\mathcal{C}$ is contractible if and only if the unique functor $\mathcal{C} \to \bullet$ is an equivalence of categories.

2.6.2

We give an example of equivalence which is a "duality", where in general, without getting into technicalities, a duality is an functor $\mathbb{D} : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ such that $\mathbb{D} \circ \mathbb{D} \cong \mathrm{Id}_{\mathcal{C}}$ (notice that the composition is defined once we realize that a functor $F : \mathcal{C} \to \mathcal{D}$ can be also considered as a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$). Let k be a field. Denote by Vec_k^{fd} the full subcategory of Vec_k consisting of the vector spaces which are finite-dimensional. We have an equivalence of categories

$$\mathbb{D} : \mathrm{Vec}_k^{fd} \xrightarrow{\approx} (\mathrm{Vec}_k^{fd})^{\mathrm{op}}$$

(we use the "$\approx$" sign to denote that the functor is an equivalence of categories), defined as follows. For $V \in \mathrm{Vec}_k^{fd}$ we take $\mathbb{D}(V)$ to be the **dual vector space** V^*. Given a morphism $T : V \to W$, the corresponding morphism between the duals is $T^* : W^* \to V^*$, given by $T^*(\zeta)(v) := \zeta(T(v))$. The functor $\mathbb{D}$ is not only an equivalence, but also satisfies the property that we have a canonical isomorphism between $\mathbb{D} \circ \mathbb{D}$ and $\mathrm{Id}_{\mathrm{Vec}_k^{fd}}$ (here there is a slight abuse of notation, in that a functor $\mathcal{C} \to \mathcal{D}^{\mathrm{op}}$ can also be considered as a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ in a very straight-forward and natural way, and the left copy of $\mathbb{D}$ in the expression $\mathbb{D} \circ \mathbb{D}$ is understood as a functor $(\mathrm{Vec}_k^{fd})^{\mathrm{op}} \to \mathrm{Vec}_k^{fd}$ following this remark).

2.6.3

Another example of a duality, a more complicated analog of the previous example, is **Pontryagin duality**. Namely, let us consider the category LCAG of locally compact abelian groups. We have an equivalence of categories

$$\mathbb{D} : \mathrm{LCAG} \xrightarrow{\approx} \mathrm{LCAG}^{\mathrm{op}},$$

which is the underpinning of the Fourier transform. Let us describe it briefly, without going into details. As a set, we let $\mathbb{D}(A)$ to be the set of continuous group homomorphisms from A to the circle group $\mathbb{R}/\mathbb{Z}$. This has the structure of an abelian group by pointwise multiplication. We give it the compact-open topology. In this way $\mathbb{D}(A)$ becomes a locally compact abelian group, one checks. We have a straight-forward functoriality (in complete analogy with the above case of dual vector spaces), given $\phi : A \to B$ constructing $\hat{\phi} : \mathbb{D}(B) \to \mathbb{D}(A)$ (by sending $\chi : B \to \mathbb{R}/\mathbb{Z}$ to $\hat{\phi}(\chi) : A \to \mathbb{R}/\mathbb{Z}$ defined by sending a to $\chi(\phi(a))$). We therefore obtain our functor $\mathbb{D}$, and need to check that it is an equivalence of categories. Again we in fact have that $\mathbb{D} \circ \mathbb{D}$ admits a canonical isomorphism with $\mathrm{Id}_{\mathrm{LCAG}}$. Namely, we construct naturally a morphism $\mathrm{Id}_{\mathrm{LCAG}} \to \mathbb{D} \circ \mathbb{D}$, and then check that it is an isomorphism. To construct this morphism, one needs to construct a morphism $A \to \mathbb{D}(\mathbb{D}(A))$ for any locally compact abelian group A. One does it by sending a to the morphism $\mathbb{D}(A) \to \mathbb{R}/\mathbb{Z}$ sending χ to $\chi(a)$. Then one checks everything (some parts are not immediate, one can consult a book on abstract harmonic analysis).

For example, Pontryagin duality sends compact groups to discrete groups and discrete groups to compact groups.

2.6.4

Now let us give an example, where the feeling is that we describe a "skeleton" of our category, omitting repetitions of objects (i.e. leaving one object from each isomorphism class). Let us denote by Mat_k the following category. The class of objects is $\mathbb{Z}_{\geq 0}$. The set of morphisms from m to n is the set $M_{n\times m}(k)$ of $n \times m$-matrices over k. Composition is via multiplication of matrices. We have a functor

$$\mathrm{Mat}_k \to \mathrm{Vec}_k^{fd}$$

given as follows. It sends n to k^n. It sends a morphism from m to n, so a matrix $A \in \mathrm{Mat}_{n\times m}(k)$, to the k-linear transformation $k^m \to k^n$ sending the standard basis vector e_i to the linear combination of standard basis vectors $\sum A_{ji} f_j$. This functor is an equivalence of categories.

2.6.5

Here is another example of "skeletal" nature. Let G be a group. Recall that a **G-set** is a pair consisting of a set X and an action map $a : G \times X \to X$, satisfying $a(g_2, a(g_1, x)) = a(g_2 g_1, x)$ and $a(1, x) = x$. One usually abbreviates and writes gx instead of $a(g, x)$, so a is implicit. A morphism of G-sets from X to Y is a map $f : X \to Y$ satisfying $f(gx) = gf(x)$. One obtains the category G-Set of G-sets. A G-set X is called a **G-torsor** if for any $x_1, x_2 \in X$ there exists a unique $g \in G$ such that $gx_1 = x_2$. We denote by G-Tors the full subcategory of G-Set consisting of G-torsors. We have a functor

$$G^{\mathrm{op}} \backslash * \to G\text{-Tors}$$

given as follows. It sends $*$ to the G-torsor G (where the action is by multiplication on the left). It sends a morphism $* \xrightarrow{g} *$ to the morphism $G \to G$ given by $x \mapsto xg$. This functor is an equivalence of categories.

2.7 Theory of the fundamental groupoid and covering spaces

2.7.1

Let $X \in \mathrm{Top}$. A set one attaches to X is the **set** $\pi_0(X)$ **of path-connected components** of X. Namely, we define an equivalence relation on X (viewed as a set), by declaring $x_1 \in X$ to be equivalent to $x_0 \in X$ if there exists a continuous map $p : [0, 1] \to X$ such that $p(0) = x_0$ and $p(1) = x_1$ (we will call such a continuous map a **path from** x_0 **to** x_1).

Exercise 2.5. *Check that this is indeed an equivalence relation.*

By definition, the set $\pi_0(X)$ of path-connected components of X is the set of equivalence classes of this equivalence relation. For example, X is called **path-connected** if $|\pi_0(X)| = 1$ (i.e. $\pi_0(X)$ is a contractible set, in the terminology discussed above).

Thus, in $\pi_0(X)$ we consider points of X "the same" if they are "connectable". But, similarly to the discussion above, one might think about remembering how points can be connected, not only whether they are connectable, and this gives rise to a groupoid, the **fundamental groupoid** $\Pi(X)$ which we will now define.

For $x_0, x_1 \in X$, let $p_0, p_1 : [0,1] \to X$ be two paths from x_0 to x_1. We say that p_1 is **(fixed-end-point) homotopic** to p_0 if these are connected by a "path of paths", namely there exists a continuous map $P : [0,1] \times [0,1] \to X$ such that $P(-,0) = x_0$, $P(-,1) = x_1$, $P(0,-) = p_0(-)$ and $P(1,-) = p_1(-)$.

Exercise 2.6. *Check that being fixed-end-point homotopic is an equivalence relation on the set of paths from x_0 to x_1.*

We now define the fundamental groupoid $\Pi(X)$ (maybe $\pi_{\leq 1}(X)$ is a better notation) as follows. The class of objects of $\Pi(X)$ is the set X. For $x_0, x_1 \in X$, the set of morphisms from x_0 to x_1 in $\Pi(X)$ is the set of equivalence classes of paths from x_0 to x_1, under the fixed-end-point homotopy equivalence relation. Composition is defined by **concatenation** of paths - if we have a path p from x_0 to x_1 and a path q from x_1 to x_2, then we define a path from x_0 to x_2 as the map $[0,1] \to X$ sending $t \in [0,1/2]$ to $p(2t)$ and $t \in [1/2,1]$ to $q(2(t-1/2))$.)

Exercise 2.7. *Check that in this way everything is well defined and we indeed obtain a groupoid.*

Remark 2.18. Notice that the sets $\pi_0(X)$ and $\pi_0(\Pi(X))$ (where the latter is the set of isomorphism classes in a groupoid, discussed above), are in natural bijection.

The space X is called **simply-connected** if $\Pi(X)$ is a contractible groupoid (as discussed above), i.e. $\Pi(X)$ is non-empty and in $\Pi(X)$, between any two objects there exists exactly one morphism. In other words, X should be non-empty and connected (so that between any two points there exists a path) and for every two points of X, and two paths from the one to the other, these paths are fixed-end-point homotopic.

Exercise 2.8. *Show that the open unit disc D_n is simply connected.*

Example 2.19. *A very basic mathematical ground fact is that the circle S^1 is not simply connected. In fact, for a point $x \in S^1$, the group $\mathrm{End}_{\Pi(X)}(x)$ is isomorphic to $\mathbb{Z}$.*

2.7.2

Given a category $\mathcal{C}$ and an object $C \in \mathcal{C}$, one define the **over-category** $\mathcal{C}_{/C}$ as follows. An object of $\mathcal{C}_{/C}$ is an object X of $\mathcal{C}$ equipped with a morphism $\alpha : X \to C$ in $\mathcal{C}$. Given two such objects (X, α) and (X', α'), a morphism from the first to the second is a morphism $\beta : X \to X'$ such that $\alpha' \circ \beta = \alpha$. Composition is straight-forward.

Given our topological space X, we will now define a full subcategory $\mathrm{Cov}(X) \subset \mathrm{Top}_{/X}$, consisting of **covering maps of** X.

2.7.3

Let us recall which morphisms $Y \to X$ in Top are called covering maps (i.e. sit by definition in $\mathrm{Cov}(X)$). Given a set F, which we consider as a discrete topological space, and a topological space X, the **trivial covering map of X with fiber F** is the morphism $F \times X \to X$ of projection on the second factor. A morphism $Y \to X$ is a **trivializable covering map** if it is isomorphic in $\mathrm{Top}_{/X}$ to a trivial covering map as above (for some F). A morphism $\pi : Y \to X$ is a **covering map** if locally on X it becomes a trivializable covering map. In other words, the condition is that for every $x \in X$ there exists an open $x \in U \subset X$ such that the morphism $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is a trivializable covering map (this now is an object in $\mathrm{Top}_{/U}$).

A basic example of covering maps is the following. Consider the circle $C \subset \mathbb{C}$ of numbers of norm 1. Given $n \in \mathbb{Z}_{\geq 1}$, we define $\pi_n : C \to C$ by $z \mapsto z^n$. Then π_n is a covering map, all of whose fibers have cardinality n. A related example is $\pi : \mathbb{R} \to C$ given by $x \mapsto e^{2\pi i x}$. It is a covering map all of whose fibers are in bijection (non-canonically) with $\mathbb{Z}$.

2.7.4

Let us also mention pull-back of covering maps. First, recall that given topological spaces X_1, X_2, X equipped with morphisms $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$, we define the **fiber product** $X_1 \underset{X}{\times} X_2$ as the subspace of $X_1 \times X_2$ consisting of (x_1, x_2) for which $p_1(x_1) = p_2(x_2)$. Now, given a morphism $f : X' \to X$ in Top and $\pi : Y \to X$ in $\mathrm{Cov}(X)$, we consider $Y' := X' \underset{X}{\times} Y$ equipped with the $\pi' : Y' \to X'$ sending (x', y) to x'. One then checks that $\pi' : Y' \to X'$ is a covering map, which we denote $f^*(\pi)$, and in this way one can naturally define a functor $f^* : \mathrm{Cov}(X) \to \mathrm{Cov}(X')$.

2.7.5

Given a morphism $\pi : Y \to X$ in a category, a **section** of π is another name for a right inverse for π, i.e. a morphism $s : X \to Y$ such that $\pi \circ s = \mathrm{id}_X$.

Claim 2.20 (Path lifting lemma)**.** *Let $\pi : Y \to [0,1]^n$ be a covering map. Then the map from the set of sections of π to the set $\pi^{-1}(0)$, given by sending s to $s(0)$, is a bijection.*

Proof. We omit the proof. □

2.7.6

Let $X \in \mathrm{Top}$. Using Claim 2.20, we can construct a functor

$$\mathrm{Act} : \mathrm{Cov}(X) \to \mathrm{Fun}(\Pi(X), \mathrm{Set})$$

as follows. Let $\pi : Y \to X$ be a covering map. We define the functor $\mathrm{Act}(\pi)$ to send $x \in X$ to $\pi^{-1}(x)$. Given a morphism $x_1 \to x_2$ in $\Pi(X)$, we need to define a map of sets $\pi^{-1}(x_1) \to \pi^{-1}(x_2)$. Let $y_1 \in \pi^{-1}(x_1)$. Choose a path $p : [0,1] \to X$ from x_1 to x_2 representing our morphism (which is a homotopy class of paths). By Claim 2.20, there exists a unique section of $p^*(\pi)$ whose value over 0 is y_1. We define our map $\pi^{-1}(x_1) \to \pi^{-1}(x_2)$ to send y_1 to the value of this section over 1. We leave to the reader to check, using Claim 2.20 for $[0,1]^2$ and some of its boundary copies of $[0,1]$, that this procedure does not depend on the choice of the path p in the homotopy class. Then one can check all the other compatibilities and conditions, verifying that indeed in this way we obtain a functor Act as desired. The main theorem is:

Theorem 2.21. *Assume the technical assumption that X is locally simply-connected. Then the functor*

$$\mathrm{Act} : \mathrm{Cov}(X) \to \mathrm{Fun}(\Pi(X), \mathrm{Set})$$

is an equivalence of categories.

For X to be locally simply-connected means that for every $x \in X$ there exists an open $x \in U \subset X$ such that U is simply-connected. This is a technical assumption - one should imagine that all spaces are locally simply-connected, in the kind of things we pursue here.

This theorem relates two objects of different "materiality", so a good example of an equivalence of categories. Indeed, $\Pi(X)$ talks about paths in X, their deformations, how freely we can travel inside X and so on. $\mathrm{Cov}(X)$ talks about the construction of coverings of X, so ways of enlarging X by creating locally several copies of it in a consistent manner (as in the example of $\pi : \mathbb{R} \to C$ above - instead of just thinking about a complex number of absolute value 1, one thinks of such a number together with a representation of it as $e^{2\pi i x}$ for some real number x).

Let us sketch the construction of a functor inverse to Act, without going into details of showing that indeed the constructions are inverse to each other (i.e. the construction of isomorphisms of their compositions with the identity functors). We construct a functor

$$\mathrm{Fun}(\Pi(X), \mathrm{Set}) \to \mathrm{Cov}(X)$$

as follows. Given $S \in \mathrm{Fun}(\Pi(X), \mathrm{Set})$, we construct a topological space Y, as a set being the disjoint union $\coprod_{x \in X} S(x)$. We consider the topology on Y generated by open sets of the following form. For open $U \subset X$, $x_0 \in U$ and $s_0 \in S(x_0)$, consider the subset of Y consisting of $S(\alpha)(s_0)$ for all $x_1 \in U$ and morphisms $\alpha : x_0 \to x_1$ in $\Pi(X)$. We have a natural "projection" map $\pi : Y \to X$ given by sending elements of $S(x)$ to x. One shows that π is continuous and is a covering map. One then naturally makes the construction functorial, obtaining the desired functor.

2.7.7

Given a groupoid $\mathcal{S}$ and an object $s \in \mathcal{S}$, the set $G := \mathrm{Hom}_{\mathcal{S}}(s,s)$ is naturally a group, using composition as multiplication. We have a functor

$$\mathrm{Fun}(\mathcal{S}, \mathrm{Set}) \to G\text{-Set},$$

sending $F : \mathcal{S} \to \mathrm{Set}$ to $F(s)$, where the action $\mathrm{Hom}_{\mathcal{S}}(s,s) \times F(s) \to F(s)$ is given by $(\alpha, \zeta) \mapsto F(\alpha)(\zeta)$. If $\mathcal{S}$ is **connected**, meaning that $|\pi_0(\mathcal{S})| = 1$, i.e. every two objects in $\mathcal{S}$ are isomorphic, then this functor is an equivalence of categories.

In the setting of Theorem 2.21, let us suppose that X is connected (let us remark here that if X is locally connected, then for X to be connected in the usual topological sense is the same as for X to be path-connected). This is equivalent to the groupoid $\Pi(X)$ being connected (as a groupoid, in the sense above). Fix a base point $x_0 \in X$. The group $\mathrm{Hom}_{\Pi(X)}(x_0, x_0)$ is called the **fundamental group of** (X, x_0), and denoted $\pi_1(X, x_0)$. Then by the said above, we have an equivalence of categories

$$\mathrm{Fun}(\Pi(X), \mathrm{Set}) \to \pi_1(X, x_0)\text{-Set}.$$

Pre-composing with our equivalence of categories Act, we obtain an equivalence of categories

$$\mathrm{Act}_{x_0} : \mathrm{Cov}(X) \to \pi_1(X, x_0)\text{-Set}.$$

2.7.8

Let us illustrate a bit using the equivalence Act_{x_0}. We fix $X \in \mathrm{Top}$ which is locally simply connected and connected, and $x_0 \in X$ a base point.

Lemma 2.22. *Let $\pi : Y \to X$ be in $\mathrm{Cov}(X)$. Then Y is connected if and only if $\mathrm{Act}_{x_0}(Y)$ is a transitive $\pi_1(X, x_0)$-set. Assuming that, Y is simply connected if and only if $\mathrm{Act}_{x_0}(Y)$ is a free $\pi_1(X, x_)$-set.*

Proof. This is left as an exercise. □

Now, since there exists a free and transitive G-set, and all free and transitive G-sets are isomorphic, using the previous lemma we obtain that there exists a covering map $Y \to X$ such that Y is simply connected, and every two such covering maps are isomorphic. One calls such a covering map (or Y) the **universal cover of** X. For example, the universal cover of the circle C that we considered above is $\mathbb{R}$, via $\pi : \mathbb{R} \to C$ that we considered above.

2.8 Affine algebraic varieties

2.8.1

Let k be an algebraically closed field, for example $\mathbb{C}$. Let us define the category Aff_k of **"concrete" affine algebraic varieties over** k. An object of this category is a pair (n, V) consisting of $n \in \mathbb{Z}_{\geq 0}$ and a subset $V \subset k^n$, such that V has the form

$$\{(c_1, \ldots c_n) \in k^n \mid \phi_1(c_1, \ldots, c_n) = 0, \ldots, \phi_m(c_1, \ldots, c_n) = 0\}$$

for some $\phi_1, \ldots, \phi_m \in k[x_1, \ldots, x_n]$ (polynomials in n variables with coefficients in k). Given objects (n_1, V_1) and (n_2, V_2), a morphism from the first to the second is a map $m : V_1 \to V_2$ for which there exist polynomials $\phi_1, \ldots, \phi_{n_2} \in k[x_1, \ldots, x_{n_1}]$ such that

$$m(c_1, \ldots c_{n_1}) = (\phi_1(c_1, \ldots, c_{n_1}), \ldots, \phi_{n_2}(c_1, \ldots, c_{n_1})) \quad \forall (c_1, \ldots, c_{n_1}) \in V_1.$$

Composition is just composition of maps.

Some objects in Aff_k "look different" but are in fact isomorphic. A very simple example is $(1, k)$ and $(2, \{(c_1, c_2) \mid c_2 = c_1\})$. We have the morphism $c \mapsto (c, c)$ in first direction and the morphism $(c_1, c_2) \mapsto c_1$ in second direction, and those are mutually inverse.

2.8.2

Recall the notion of a **commutative k-algebra** - a k-vector space A equipped with a k-bilinear multiplication map $A \times A \to A$, which is associative, commutative and with 1. Morphisms of k-algebras are defined in the obvious way, and we obtain the category CAlg_k of commutative k-algebras.

The basic example of a commutative k-algebra is $k[x_1, \ldots, x_n]$, the k-algebra of polynomials in n variables with coefficients in k. It has the property that for a commutative k-algebra A, the map

$$\mathrm{Hom}_{\mathrm{CAlg}_k}(k[x_1, \ldots, x_n], A) \to A^n : \quad \phi \mapsto (\phi(x_1), \ldots, \phi(x_n))$$

is a bijection.

Given a k-algebra A, a **sub k-algebra** of A is a subset $B \subset A$ which is a sub k-vector space, such that $1 \in B$ and such that $a_1 a_2 \in B$ for $a_1, a_2 \in B$.

Let A be a commutative k-algebra. Let $a_1, \ldots, a_n \in A$. There exists a unique k-algebra morphism $k[x_1, \ldots, x_n] \to A$ sending x_i to a_i; namely, it sends a polynomial $\phi(x_1, \ldots, x_n)$ to the evaluation $\phi(a_1, \ldots, a_n)$. The image of this morphism is a sub k-algebra of A, containing the elements $a_1, \ldots, a_n$ and contained in any other sub k-algebra of A containing these elements. We say that $a_1, \ldots, a_n$ **generate** A (as a k-algebra) if this smallest sub k-algebra containing these elements is the whole A; equivalently if the morphism $k[x_1, \ldots, x_n] \to A$ mentioned is surjective. We say that A is **finitely generated** (as a k-algebra) if there exist $a_1, \ldots, a_n \in A$ which generate A. Equivalently, if there exists a surjective k-algebra morphism from some $k[x_1, \ldots, x_n]$ to A.

Given a commutative k-algebra A, recall that an **ideal** in A is a subset $I \subset A$ which is a sub k-vector space, and such that $ai \in I$ for $a \in A$ and $i \in I$. Given $i_1, \ldots, i_n \in A$, one defines an ideal

$$(i_1, \ldots, i_n)_A := \{a_1 i_1 + \ldots + a_n i_n : \ a_1, \ldots, a_n \in A\}.$$

This is an ideal containing the elements $i_1, \ldots, i_n$ and contained in any other ideal containing these elements. Given an ideal $I \subset A$, we say that I is **finitely generated** if there exist $i_1, \ldots, i_n \in A$ such that $I = (i_1, \ldots, i_n)_A$.

Proposition 2.23. *Let A be a finitely generated commutative k-algebra. Then every ideal in A is finitely generated.*

Given a commutative k-algebra A and an ideal $I \subset A$, one can form the k-algebra A/I, the **quotient k-algebra**. Its elements are cosets $a + I := \{a + i : i \in I\}$ in A, and all operation are done using choices of representatives (and then showing that the result does not depend on the choices). One has the **isomorphism theorem**: Given a surjective morphism of commutative k-algebras $t : B \to A$, the kernel $\mathrm{Ker}(t)$ is an ideal in B, and we have an isomorphism of commutative k-algebras $B/\mathrm{Ker}(t) \xrightarrow{\sim} A$ given by sending $b + \mathrm{Ker}(t)$ to $t(b)$.

2.8.3

Let $(n, V) \in \mathrm{Aff}_k$. The set of morphisms from (n, V) to $(1, k)$ is called the set of **regular (or algebraic) functions on** (n, V). So those are functions $f : V \to k$ for which there exists a polynomial $\phi \in k[x_1, \ldots, x_n]$ such that $f(c_1, \ldots, c_n) = \phi(c_1, \ldots, c_n)$ for all $(c_1, \ldots, c_n) \in V$. Notice that the set of regular function on (n, V) has a natural structure of k-algebra, by pointwise addition, multiplication and multiplication by scalar. Let us denote this k-algebra of regular function on (n, V) by $\mathcal{O}(n, V)$. The assignment $(n, V) \mapsto \mathcal{O}(n, V)$ can be naturally made into a functor

$$(\mathrm{Aff}_k)^{\mathrm{op}} \to \mathrm{CAlg}_k,$$

since we can pull back regular functions.

It is easy to find some conditions which objects in the essential image of this functor must obey. Let $(n, V) \in \mathrm{Aff}_k$. We have a map $k[x_1, \ldots, x_n] \to \mathcal{O}(n, V)$, given simply by considering the function on V defined by a polynomial. Clearly, this map is a morphism of k-algebras. By the definition of $\mathcal{O}(n, V)$, it is surjective. Thus, first of all, the k-algebra $\mathcal{O}(n, V)$ is **finitely generated**. Second, this k-algebra is **reduced**, meaning that it has no nilpotent elements except 0. Indeed, this is an algebra of k-valued functions under pointwise multiplication, and so it being reduced immediately follows from k being reduced, which is clear as it is a field.

Let us denote by $\mathrm{CAlg}_k^{fg,red} \subset \mathrm{CAlg}_k$ the full subcategory consisting of finitely generated and reduced k-algebras. We therefore have a functor

$$\mathcal{O} : (\mathrm{Aff}_k)^{\mathrm{op}} \to \mathrm{CAlg}_k^{fg,red} : \quad (n, V) \mapsto \mathcal{O}(n, V).$$

Theorem 2.24 (the basic theorem of basic algebraic geometry)**.** *The functor $\mathcal{O}$ above is an equivalence of categories.*

Proof. It is quite easy to see that $\mathcal{O}$ is fully faithful (we leave this as an exercise). That $\mathcal{O}$ is essentially surjective follows from **Hilbert's nullstellensatz**:

Theorem 2.25 (Hilbert's nullstellensatz). *Let $I \subset k[x_1, \ldots, x_n]$ be an ideal. Let $\phi \in k[x_1, \ldots, x_n]$ be a polynomial such that $\phi(c_1, \ldots, c_n) = 0$ for every $(c_1, \ldots, c_n) \in k^n$ for which $\psi(c_1, \ldots, c_n) = 0$ for all $\psi \in I$. Then for some $m \in \mathbb{Z}_{\geq 1}$ one has $\phi^m \in I$.*

Now, given $A \in \mathrm{CAlg}_k^{fg,red}$, choose a surjective k-algebra morhism $k[x_1, \ldots, x_n] \to A$. Denote by I its kernel. Then A is isomorphic to $k[x_1, \ldots, x_n]/I$, so by definition it is enough to see that $k[x_1, \ldots, x_n]/I$ lies in the image of $\mathcal{O}$. Consider $V \subset k^n$ given by

$$V := \{(c_1, \ldots, c_n) \in k^n \mid \psi(c_1, \ldots, c_n) = 0 \ \forall \psi \in I\}.$$

In fact, recall that I is a finitely generated ideal, so there exist $\psi_1, \ldots, \psi_m \in k[x_1, \ldots, x_n]$ such that $I = (\psi_1, \ldots, \psi_m)_{k[x_1, \ldots, x_n]}$, and it is immediate to see that

$$V = \{(c_1, \ldots, c_n) \in k^n \mid \psi_1(c_1, \ldots, c_n) = 0, \ldots, \psi_m(c_1, \ldots, c_n) = 0\},$$

so that (n, V) is an affine algebraic variety. Now, we consider the morphism of k-algebras $k[x_1, \ldots, x_n] \to \mathcal{O}(n, V)$ given by sending a polynomial to the function it defines on V by evaluation. As mentioned above, this is surjective. The kernel consists of polynomials $\phi \in k[x_1, \ldots, x_n]$ for which $\phi(c_1, \ldots, c_n) = 0$ for all $(c_1, \ldots, c_n) \in V$. By Hilbert's nullstellensatz we see that the kernel consists of polynomials ϕ for which $\phi^m \in I$ for some $m \in \mathbb{Z}_{\geq 1}$. But, since A is reduced, so is the isomorphic to it $k[x_1, \ldots, x_n]/I$, and this translates by definitions to the property that for $\phi \in k[x_1, \ldots, x_n]$ for which $\phi^m \in I$ for some $m \in \mathbb{Z}_{\geq 1}$, we have $\phi \in I$. In other words, we see that the kernel of our morphism $k[x_1, \ldots, x_n] \to \mathcal{O}(n, V)$ is precisely I. Hence $\mathcal{O}(n, V)$ is isomorphic to $k[x_1, \ldots, x_n]/I$, which is isomorphic to A.

□

We interpret the above equivalence of categories between $(\mathrm{Aff}_k)^{\mathrm{op}}$ and $\mathrm{CAlg}_k^{fg,red}$ as translating the study of geometric objects into pure algebra. Later, Grothendieck created a reverse process, where one studies commutative rings which a-priori seem to be distant from geometry, such as the ring of integers $\mathbb{Z}$, by finding geometric objects on which these rings are the rings of functions, in some sense (this is the theory of **affine schemes**) - so establishing an equivalence of categories between $\mathrm{AffineSchemes}^{\mathrm{op}}$ and CRng. In yet later treatments, when category theory is absorbed even more, one arrives to simply defining the category of affine schemes as $\mathrm{CRng}^{\mathrm{op}}$ (and we wonder, where is the geometry then?).

2.9 The Gelfand transform

It is also tempting to describe the equivalence of categories between the category of commutative unital C^*-algebras and the category opposite to that of compact Hausdorff spaces. But we are not sure the generic reader has enough background (such as spectral study of bounded operators on a Hilbert space) to have motivation for that statement, so we leave this for now.

2.10 Galois theory

Another good example of an equivalence of categories, which we also omit for now, is Galois theory (given a field K, choosing an algebraic closure of it and thus obtaining a Galois group Γ (which carries a profinite topology), we have an equivalence between the opposite of the category of finite-dimensional semisimple K-algebras and the category of continuous finite Γ-sets).

3 Yoneda's lemma, representing objects, limits

3.1 Yoneda's lemma

We will now give the most important example of a fully faithful functor, the contractibility of fibers of which serves a very important ideology.

3.1.1

Let $\mathcal{C}$ be a category. We consider the category of functors $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ (it is also called the category of **presheaves** on $\mathcal{C}$ for reasons which are currently not relevant for us). Let us note that functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ are also called **contravariant functors from $\mathcal{C}$ to $\mathcal{D}$**.

Any object $c \in \mathcal{C}$ provides a functor $\mathrm{Ynd}_{\mathcal{C}}(c) \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ as follows. We set $\mathrm{Ynd}_{\mathcal{C}}(c)(x) := \mathrm{Hom}_{\mathcal{C}}(x, c)$. Given a morphism $\beta : x_1 \to x_2$ we have the map

$$\mathrm{Ynd}_{\mathcal{C}}(c)(x_2) = \mathrm{Hom}_{\mathcal{C}}(x_2, c) \xrightarrow{-\circ\beta} \mathrm{Hom}_{\mathcal{C}}(x_1, c) = \mathrm{Ynd}_{\mathcal{C}}(c)(x_1).$$

We therefore think of functors $F \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ as "imaginary objects" of $\mathcal{C}$. Namely, following the pattern of the above example $\mathrm{Ynd}_{\mathcal{C}}(c)$, we think of $F(x)$ as the set of morphisms from x to F, and given $\beta : x_1 \to x_2$ we think of the corresponding $F(x_1) \to F(x_2)$ as sending a morphism from x_1 to F to its pre-composition with β, yielding a morphism from x_2 to F.

Now, the association $c \mapsto \mathrm{Ynd}_{\mathcal{C}}(c)$ above can be made in a straight-forward way into a functor $\mathrm{Ynd}_{\mathcal{C}} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$. Namely, to a morphism $\beta : c_1 \to c_2$ we associate the morphism $\mathrm{Ynd}_{\mathcal{C}}(c_1) \to \mathrm{Ynd}_{\mathcal{C}}(c_2)$ which for every $x \in \mathcal{C}$ is specified by $\mathrm{Ynd}_{\mathcal{C}}(c_1)(x) \to \mathrm{Ynd}_{\mathcal{C}}(c_2)(x)$ given by

$$\mathrm{Ynd}_{\mathcal{C}}(c_1)(x) = \mathrm{Hom}_{\mathcal{C}}(x, c_1) \xrightarrow{\beta\circ-} \mathrm{Hom}_{\mathcal{C}}(x, c_2) = \mathrm{Ynd}_{\mathcal{C}}(c_2)(x).$$

We call this functor

$$\mathrm{Ynd}_{\mathcal{C}} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$$

the **Yoneda embedding**.

3.1.2

Here is a precise formulation stemming from thinking about $F(c)$ as the set of morphisms from c to F:

Lemma 3.1. *Let $F \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ and let $c \in \mathcal{C}$. The map*

$$\mathrm{Hom}(\mathrm{Ynd}_{\mathcal{C}}(c), F) \to F(c): \quad \alpha \mapsto \alpha_c(\mathrm{id}_c)$$

is a bijection, whose inverse is the map

$$F(c) \to \mathrm{Hom}(\mathrm{Ynd}_{\mathcal{C}}(c), F)$$

sending $\zeta \in F(c)$ to $\alpha_\zeta \in \mathrm{Hom}(\mathrm{Ynd}_{\mathcal{C}}(c), F)$ given by taking $(\alpha_\zeta)_x : \mathrm{Hom}_{\mathcal{C}}(x, c) \to F(x)$ to send $\beta \in \mathrm{Hom}_{\mathcal{C}}(x, c)$ to $F(\beta)(\zeta)$.

Proof. Let us check the functoriality of the collection $((\alpha_\zeta)_x)_{x \in \mathcal{C}}$, i.e. verify that it gives a morphism of functors α_ζ from $\mathrm{Ynd}_{\mathcal{C}}(c)$ to F. So given $\gamma : x_1 \to x_2$ we need to check that the diagram

$$\begin{array}{ccc} \mathrm{Hom}(x_2, c) & \xrightarrow{(\alpha_\zeta)_{x_2}} & F(x_2) \\ {\scriptstyle -\circ\gamma}\downarrow & & \downarrow{\scriptstyle F(\gamma)} \\ \mathrm{Hom}(x_1, c) & \xrightarrow{(\alpha_\zeta)_{x_1}} & F(x_1) \end{array}$$

commutes. Going right and then down sends $\beta : x_2 \to c$ to $F(\gamma)(F(\beta)(\zeta))$. Going down and then right sends $\beta : x_2 \to c$ to $F(\beta \circ \gamma)(\zeta)$, and those are the same.

Now one has to check that the two maps are inverse to each other. First, given $\zeta \in F(c)$, we map to the morphism α_ζ, and then map back to $(\alpha_\zeta)_c(\mathrm{id}_c) = F(\mathrm{id}_c)(\zeta) = \zeta$. Second, given $\alpha : \mathrm{Ynd}_{\mathcal{C}}(c) \to F$, we map to $\zeta := \alpha_c(\mathrm{id}_c) \in F(c)$, and then map back to α_ζ, and we need to check that $\alpha = \alpha_\zeta$. So for every $x \in \mathcal{C}$ we need to check that $\alpha_x, (\alpha_\zeta)_x : \mathrm{Hom}(x, c) \to F(x)$ are equal. So for $\beta : x \to c$ we need to check that $(\alpha_\zeta)_x(\beta)$ is equal to $\alpha_x(\beta)$. By definitions, it is equal to $F(\beta)(\zeta)$, so we need to check that $\alpha_x(\beta) = F(\beta)(\alpha_c(\mathrm{id}_c))$. Considering the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(c, c) & \xrightarrow{\alpha_c} & F(c) \\ {\scriptstyle -\circ\beta}\downarrow & & \downarrow{\scriptstyle F(\beta)} \\ \mathrm{Hom}(x, c) & \xrightarrow{\alpha_x} & F(x) \end{array} \quad ,$$

we track where id_c goes. Going right and then down, it goes to $F(\beta)(\alpha_c(\mathrm{id}_c))$. Going down and then right, it goes to $\alpha_x(\beta)$. So those are equal as desired. $\square$

The word "embedding" in the name "Yoneda embedding" is justified by the following lemma, which is known for being very trivial technically but very far-reaching ideologically:

Lemma 3.2 (Yoneda's lemma)**.** *The functor*

$$\mathrm{Ynd}_{\mathcal{C}} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$$

is fully faithful.

Proof. Let $c_1, c_2 \in \mathcal{C}$. We want to show that the map

$$\mathrm{Hom}(c_1, c_2) \to \mathrm{Hom}(\mathrm{Ynd}_{\mathcal{C}}(c_1), \mathrm{Ynd}_{\mathcal{C}}(c_2))$$

given by $\mathrm{Ynd}_{\mathcal{C}}$ is bijective. Applying Lemma 3.1 to c being c_1 and F being $\mathrm{Ynd}_{\mathcal{C}}(c_2)$, we have that the map

$$\mathrm{Hom}(\mathrm{Ynd}_{\mathcal{C}}(c_1), \mathrm{Ynd}_{\mathcal{C}}(c_2)) \to \mathrm{Ynd}_{\mathcal{C}}(c_2)(c_1) = \mathrm{Hom}(c_1, c_2)$$

given by $\alpha \mapsto \alpha_{c_1}(\mathrm{id}_{c_1})$ is a bijection. So it is enough to show that the former map is right inverse to the latter map. Let $\beta : c_1 \to c_2$. The former map associated to it the morphism $\alpha : \mathrm{Ynd}_{\mathcal{C}}(c_1) \to \mathrm{Ynd}_{\mathcal{C}}(c_2)$ for which $\alpha_c : \mathrm{Hom}(c, c_1) \to \mathrm{Hom}(c, c_2)$ is given by sending a morphism $c \to c_1$ to its composition with α. So $\alpha_{c_1}(\mathrm{id}_{c_1})$, which is the result of applying the latter map, is $\alpha \circ \mathrm{id}_{c_1} = \alpha$. We are done. □

3.2 Representing objects

3.2.1

As the Yoneda embedding

$$\mathrm{Ynd}_{\mathcal{C}} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$$

is fully faithful, by Lemma 2.9 given $F \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ which lies in the essential image of $\mathrm{Ynd}_{\mathcal{C}}$, the fiber $\mathrm{Ynd}_{\mathcal{C}}^{-1}(F)$ is contractible. What does it mean concretely? It means first of all that there exists $c \in \mathcal{C}$ and an isomorphism $\alpha : \mathrm{Ynd}_{\mathcal{C}}(c) \xrightarrow{\sim} F$ (this however is simply by definition of being in the essential image). In other words, we are given for every $x \in \mathcal{C}$ a bijection

$$\alpha_x : \mathrm{Hom}_{\mathcal{C}}(x, c) \xrightarrow{\sim} F(x)$$

and it is functorial in x, meaning that given $\beta : x_1 \to x_2$ in $\mathcal{C}$ the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(x_2, c) & \xrightarrow{\alpha_{x_2}} & F(x_2) \\ \downarrow{\scriptstyle -\circ\beta} & & \downarrow{\scriptstyle F(\beta)} \\ \mathrm{Hom}_{\mathcal{C}}(x_1, c) & \xrightarrow{\alpha_{x_1}} & F(x_1) \end{array} .$$

Second, the object c is **unique, up to unique isomorphism**, in the following sense. Given another $c' \in \mathcal{C}$ with an isomoprhism $\alpha' : \mathrm{Ynd}_{\mathcal{C}}(c') \xrightarrow{\sim} F$, there exists a unique isomorphism $\epsilon : c \to c'$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Ynd}_{\mathcal{C}}(c) & \xrightarrow{\mathrm{Ynd}_{\mathcal{C}}(\epsilon)} & \mathrm{Ynd}_{\mathcal{C}}(c') \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\alpha'} & \\ & F & \end{array}$$

3.2.2

The above is the technical underpinning of a very important philosophy, pioneered (to our knowledge) by Grothendieck. Namely, in a problem one encounters one first constructs an "imaginary object" one desires, and then studies whether it is "real" - no need to verify its "realness" straight away. The jargon is that functors in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ which sit in the essential image of the Yoneda embedding are called **representable**. An object whose image under the Yoneda embedding is isomorphic to the functor at question, together with this isomorphism (i.e. an object in the relevant fiber of the Yoneda embedding) is then said to be a **representing object**. We explained above in what sense such an object is unique, up to a unique isomorphism.

Remark 3.3. Let us reiterate that to say that an object c represents the functor F means that we carry with us an isomorphism $\mathrm{Ynd}_{\mathcal{C}}(c) \xrightarrow{\sim} F$, not just the knowledge that such isomorphism exists.

3.2.3

Let us give a simple example of a representing object. Consider the functor

$$S : \mathrm{Set}^{\mathrm{op}} \to \mathrm{Set},$$

sending a set X to the set $S(X)$ of subsets of X; if we have a map $f : X \to Y$ we define a map $S(f) : S(Y) \to S(X)$ by sending a subset $A \subset Y$ to the subset $f^{-1}(A) \subset X$. The functor S is representable; indeed, consider $B := \{0, 1\}$ and consider the isomorphism of functors

$$\alpha : \mathrm{Ynd}_{\mathrm{Set}}(B) \to S$$

given by setting, for every $X \in \mathrm{Set}$,

$$\alpha_X : \mathrm{Hom}(X, B) \to S(X)$$

to be $\alpha_X(f) := f^{-1}(1)$. We leave to the reader to check that indeed in this way we obtain an isomorphism of functors α. Informally, one says "to give a subset of a set is the same as to give a map from this set to the set $\{0, 1\}$".

3.2.4

Let us give another reformulation of what it takes to have a representing object. We can of course reformulate having an object $c \in \mathcal{C}$ and an isomorphism $\mathrm{Ynd}_{\mathcal{C}}(c) \xrightarrow{\sim} F$, as having an object $c \in \mathcal{C}$ and a morphism $\alpha : \mathrm{Ynd}_{\mathcal{C}}(c) \to F$, which furthermore happens to be an isomorphism. By Lemma 3.1, the information of α is the same as that of an element $\zeta \in F(c)$. To say that α is an isomorphism is to say that for every $x \in \mathcal{C}$, the map $\mathrm{Hom}_{\mathcal{C}}(x, c) \to F(x)$ given by $\beta \mapsto \alpha(\beta)$ is a bijection. Recall from Lemma 3.1 that we have $\alpha(\beta) = F(\beta)(\zeta)$. Thus, we can reformulate that a representing object for F is an object $c \in \mathcal{C}$ equipped with an element $\zeta \in F(c)$, such that for every object $x \in \mathcal{C}$ and every $\xi \in F(x)$, there exists a unique $\beta : x \to c$ such that $\xi = F(\beta)(\zeta)$. One usually interprets this condition by saying that the pair (c, ζ) furnishes a "**universal**" element (among the elements lying in various $F(x)$'s), or that is satisfies a **universal property**.

3.2.5

Returning to the example of S and B we just had, to formulate it in the language of a universal property, we have the element $\{1\} \in S(B)$. Then $(B, \{1\})$ provides the data of an object B in Set and a universal element in $S(B)$ which, as we just explained, is an equivalent way of describing an isomorphism $\alpha : \mathrm{Ynd}_{\mathrm{Set}}(B) \to S$. Informally, $\{1\} \subset \{0, 1\}$ is a "universal" example of a subset of a set. Formally, let us unfold again, this means that for every set X and every subset $A \subset X$, there exists a unique map $f : X \to \{0, 1\}$ such that $A = f^{-1}(\{1\})$.

3.2.6

We will give a lot more examples of representing functors when we discuss limits below. For now, let us provide a non-trivial example from topology, illustrating a very important class of examples, that of **moduli problems**.

Given $X \in \mathrm{Top}$, let us consider the full subcategory $\mathrm{Cov}_2(X)$ of the category $\mathrm{Cov}(X)$, consisting of those $\pi : Y \to X$ for which $|\pi^{-1}(x)| = 2$ for all $x \in X$.

We first consider the functor

$$H : \mathrm{Top}^{\mathrm{op}} \to \mathrm{Set},$$

given by sending X to $\pi_0(\mathrm{Cov}_2(X))$, i.e. the set of isomorphism classes of 2-covering maps of X. This is functorial by pull-back of covering maps. The following claim shows that it is not reasonable to expect this functor to be representable. Recall that two morphisms of topological spaces $f_0, f_1 : X' \to X$ are said to be **homotopic** if there exists a morphism $f : [0, 1] \times X' \to X$ such that $f(0, -) = f_0(-)$ and $f(1, -) = f_1(-)$.

Claim 3.4. *Let $X, X' \in \mathrm{Top}$, and let $f_0, f_1 : X' \to X$ be two homotopic morphisms. Let $\pi : Y \to X$ be in $\mathrm{Cov}(X)$. Then $f_0^*(\pi), f_1^*(\pi) \in \mathrm{Cov}(X')$ are isomorphic.*

Proof. Let us fix a homotopy $f : [0,1] \times X' \to X$ between f_0 and f_1. Let us denote by $i_0, i_1 : X' \to [0,1] \times X'$ the morphisms $i_0(-) := (0, -)$ and $i_1(-) = (1, -)$, so that by definition $f \circ i_0 = f_0$ and $f \circ i_1 = f_1$. We want to define a map $X' \underset{f_0;X;\pi}{\times} Y \to X' \underset{f_1;X;\pi}{\times} Y$ and then check that it is gives isomorphism as desired. So we are given $y_0 \in Y$ and $x' \in X'$ such that $\pi(y_0) = f_0(x')$ and we want to define $y_1 \in Y$ such that $\pi(y_1) = f_1(x')$. We consider the path $p : [0,1] \to X$ given by $p(t) := f(t, x')$. By the path lifting lemma, it lifts uniquely to a path in Y, whose value at 0 is y_0. The value at 1 we call y_1. We omit all the checks one needs to do. $\square$

Thus, the claim hints to consider the following category HoTop, the **homotopy category of topological spaces**. Objects of HoTop are the same as objects of Top. As for morphisms, we let $\mathrm{Hom}_{\mathrm{HoTop}}(X', X)$ to be the set of equivalence classes in $\mathrm{Hom}_{\mathrm{Top}}(X', X)$ with respect to homotopy (which one checks is indeed an equivalence relation). One defines composition by choosing representatives and performing the composition in Top, checking that this does not depend on the choice of representatives.

Now instead of H, we consider the functor

$$H' : \mathrm{HoTop}^{\mathrm{op}} \to \mathrm{Set}$$

given by the same prescription as H (the claim above shows that this is well defined).

In fact, for technical reasons it is not good to work with all topological spaces, as some are pathological. Let us (in a somewhat ad-hoc manner) consider the full subcategories $\mathrm{Top}^{lsc,pc} \subset \mathrm{Top}$ and $\mathrm{HoTop}^{lsc,pc} \subset \mathrm{HoTop}$ consisting of topological spaces which are locally simply connected and paracompact. We consider the restriction of our functor to that full subcategory

$$H' : (\mathrm{HoTop}^{lsc,pc})^{\mathrm{op}} \to \mathrm{Set}$$

(by abuse of notation denoted by the same name).

Theorem 3.5. *The above functor*

$$H' : (\mathrm{HoTop}^{lsc,pc})^{\mathrm{op}} \to \mathrm{Set}$$

is representable. The representing object is called the ***classifying space of the group*** *C_2 (where C_2 is the group with two elements), or the* ***Eilenberg-Mac Lane space*** *$K(C_2, 1)$. The representing object can be concretely described (see below).*

In other words, one can find $B \in \mathrm{Top}^{lsc,pc}$ and a 2-cover $\pi : E \to B$, such that for every $X \in \mathrm{Top}^{lsc,pc}$ and every $\sigma \in \mathrm{Cov}_2(X)$, there exists a morphism of topological spaces $f : X \to B$, unique up to homotopy, such that $f^*(\pi)$ is isomorphic to σ. Let us describe, without proof, a realization of such $\pi : E \to B$. We consider the $\mathbb{R}$-vector space V with infinite basis $e_1, e_2, \ldots$. We let B be the set of one-dimensional subspaces of V. It has a natural topology whose description we omit for brevity. We let E be the set of elements in V whose length (in the usual sense of sum of squares of coordinates) is 1. It is also naturally topologized. We have the morphism $\pi : E \to B$ sending an vector to the one-dimensional subspace spanned by it. One can see that $\pi \in \mathrm{Cov}_2(B)$ and it is universal as desired.

3.2.7

Working with $\mathcal{C}^{\mathrm{op}}$ instead of $\mathcal{C}$, we obtain dual notions of corepresentable functors $\mathcal{C} \to \mathrm{Set}$, corepresenting objects, etc. Let us give an important example of corepresenting objects. Let k be a field. Let $V, W \in \mathrm{Vec}_k$. We consider the functor $B_{V,W} : \mathrm{Vec}_k \to \mathrm{Set}$ given by setting $B_{V,W}(U)$ to be the set of k-bilinear maps $V \times W \to U$. Functoriality is straight-forward. The functor $B_{V,W}$ is corepresentable, and the corepresenting object is denoted $V \underset{k}{\otimes} W$, called the **tensor product**. Thus, we have by definition bijections

$$\mathrm{Hom}_{\mathrm{Vec}_k}(V \underset{k}{\otimes} W, U) \xrightarrow{\sim} \{k\text{-bilinear maps } V \times W \to U\}$$

functorial in $U \in \mathrm{Vec}_k$. How to show the existence of $V \underset{k}{\otimes} W$? One way is as follows. Choose a basis $(e_i)_{i \in I}$ for V and a basis $(f_j)_{j \in J}$ for W. Consider the vector space $k[I \times J]$, with a basis consisting of formally created symbols $\delta_{i,j}$, one for each $(i,j) \in I \times J$. Given a k-bilinear map $\Phi : V \times W \to U$, we construct a k-linear map $k[I \times J] \to U$, given by sending $\sum_{(i,j) \in I \times J} c_{i,j} \cdot \delta_{i,j}$ to $\sum_{(i,j) \in I \times J} c_{i,j} \cdot \Phi(e_i, f_j)$. One checks that this gives a bijection

$$\mathrm{Hom}_{\mathrm{Vec}_k}(k[I \times J], U) \xleftarrow{\sim} \{k\text{-bilinear maps } V \times W \to U\}$$

functorial in U, as desired. A different approach of constructing $V \underset{k}{\otimes} W$, which will generalize better, is to consider the "huge" vector space $k[V \times W]$ (with basis element $\delta_{v,w}$ for every $(v,w) \in V \times W$), and quotient it by the subspace spanned by vectors of the form $\delta_{v_1+v_2,w} - \delta_{v_1,w} - \delta_{v_2,w}$, $\delta_{cv,w} - c\delta_{v,w}$, and two similar expressions, swapping v and w. Again, given a bilinear map $\Phi : V \times W \to U$, we obtain a linear map from this quotient space by sending $\delta_{v,w}$ to $\Phi(v,w)$, and checking that everything is well-defined. Then one checks that this is bijective and functorial.

It is very important to understand the ideology, that we don't need a concrete construction of an object to know what it is. The construction is simply a proof that the tensor product exists. One might be able to prove it more abstractly, without needing a concrete construction. One hopes that when working with the tensor product, one will be able, mostly, to be satisfied by its universal property, without needing to remember a concrete construction. Or, at least, one tries that all the statements that one makes do not address a concrete construction, while proofs sometimes will.

Still on the ideological level, somewhat dually to the previous paragraph, we can say that it can be useful to know of a construction/"realization"/"model" of a representing object, rather than to know the mere fact of its existence. Moreover, if we know one such "model", it still might be useful to learn another one (like in the example above of the tensor product, where we had two descriptions of it). Each "model" can shed light on some other aspect. Moreover, the comparison of two "models" (i.e. expliciting the isomorphism between them) can also shed light on some things!

3.3 The definition of a limit

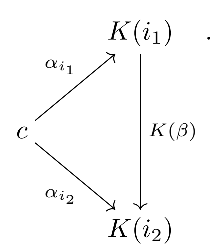
3.3.1

Let $\mathcal{I}$ be a category, of which we think as an "index category". Of a functor $\mathcal{I} \to \mathcal{C}$ we think as a "diagram in $\mathcal{C}$ of shape $\mathcal{I}$".

For example, if $\mathcal{I} = *$ then to give a diagram in $\mathcal{C}$ of shape $\mathcal{I}$ is the same as to give an object of $\mathcal{C}$. If $\mathcal{I} = \{*_1, *_2\}$ is a set with two elements (considered as a category with only identity morphisms), then to give a diagram in $\mathcal{C}$ of shape $\mathcal{I}$ is the same as to give an ordered pair of objects of $\mathcal{C}$. Of course, generalizing this, if $\mathcal{I}$ is a set (considered as a category with only identity morphisms) then to give a diagram in $\mathcal{C}$ of shape $\mathcal{I}$ is the same as to give an object of $\mathcal{C}$ for each element in $\mathcal{I}$. If $\mathcal{I}$ is the category having objects $*_1, *_2$, and whose only non-identity morphism is one morphism from $*_1$ to $*_2$, then to give a diagram in $\mathcal{C}$ of shape $\mathcal{I}$ is the same as to give a morphism in $\mathcal{C}$.

3.3.2

Let us be given $K : \mathcal{I} \to \mathcal{C}$. Let $c \in \mathcal{C}$. By a **cone with vertex c over K** we will understand the data of a morphism $\alpha_i : c \to K(i)$ for every $i \in \mathcal{I}$, such that for any morphism $\beta : i_1 \to i_2$ in $\mathcal{I}$ the following diagram is commutative:



The set of such data we denote $\mathrm{Cones}_K(c)$.

3.3.3

We naturally obtain a functor $\mathrm{Cones}_K \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ (i.e. the reader should explicit what it does to morphisms etc.). If this functor is representable, a representing object is called the **limit** of K, and denoted $\lim K$, or $\lim_{i \in I} K(i)$.

3.3.4

According to what we explained above, a reformulation of the notion of the limit of K is that it is an object $c \in \mathcal{C}$ equipped with a cone $(\alpha_i : c \to K(i))_{i \in \mathcal{I}}$ with vertex c over K (we will refer to α_i as the **structural morphisms**, or **evaluations/projections** of the limit) which is universal: If $c' \in \mathcal{C}$ is an object equipped with a cone $(\beta_i : c' \to K(i))_{i \in \mathcal{I}}$ with vertex c' over K, there exists a unique morphism $\gamma : c' \to c$ such that $\beta_i = \alpha_i \circ \gamma$ for all $i \in \mathcal{I}$.

3.4 Examples of limits

3.4.1

Let us now consider $\mathcal{C}$ to be some typical relatively concrete category, like Set or AbGrp or Vec_k or CRng etc.; Say $\mathcal{C} := \mathrm{CRng}$. Assume that $\mathcal{I}$ is a **small category** (i.e. the Hom-sets are all small, and the class of objects of $\mathcal{I}$ is a small set). Let us describe then completely explicitly what is $\lim K$. We have the commutative ring $\prod_{i \in \mathcal{I}} K(i)$, whose elements are vectors $(k_i)_{i \in \mathcal{I}}$ where $k_i \in K(i)$, and the sum and product are defined component-wise. We then consider the subring A of $\prod_{i \in \mathcal{I}} K(i)$, consisting of vectors $(k_i)_{i \in \mathcal{I}}$ such that for every morphism $\alpha : i \to j$ in $\mathcal{I}$, one has $K(\alpha)(k_i) = k_j$. We have ring morphsims $\mathrm{pr}_i : A \to K(i)$ for every $i \in \mathcal{I}$, given by projecting onto the i-th coordinate. These are clearly compatible, forming a cone with vertex A over K. We then claim that this makes A to be $\lim K$. For this, one needs to check that given a commutative ring B and a morphisms $\beta_i : B \to K(i)$ for $i \in \mathcal{I}$, compatible, one has a unique morphism $\beta : B \to A$ such that $\mathrm{pr}_i \circ \beta = \beta_i$ for $i \in \mathcal{I}$. Clearly $\beta(b) := (\beta_i(b))_{i \in \mathcal{I}}$ will be the unique morphisms as desired - we leave to the reader to check this.

3.4.2

For $\mathcal{I} = \emptyset$ (so $K : \mathcal{I} \to \mathcal{C}$ is clear) we obtain the notion of the final object in $\mathcal{C}$. By definition, this is an object $c \in \mathcal{C}$ equipped with a bijection

$$\mathrm{Hom}_{\mathcal{C}}(c', c) \xrightarrow{\sim} *$$

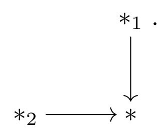
for every c' (because there exists a unique cone with vertex c' over K - the empty one). Notice that functoriality in c' here is automatic. So, in other words, a final object is an object such that from any object there exists a unique morphism into it.

3.4.3

More generally, for $\mathcal{I}$ a set considered as a category with only identity morphisms, and a diagram $K : \mathcal{I} \to \mathcal{C}$, the limit $\lim K$, in this case commonly denoted $\prod_{i \in \mathcal{I}} K(i)$ and called the **product** of the $K(i)$'s, is an object equipped with "projection" morphisms $\pi_i : \prod_{i \in \mathcal{I}} K(i) \to K(i)$ for every $i \in \mathcal{I}$, with the property that for every object $c \in \mathcal{C}$ equipped with morphisms $\beta_i : c \to K(i)$ for all $i \in \mathcal{I}$, there exists a unique morphism $\beta : c \to \prod_{i \in \mathcal{I}} K(i)$ such that $\pi_i \circ \beta = \beta_i$ for all $i \in \mathcal{I}$. A common particular case is $\mathcal{I} = \{*_1, *_2\}$. Then K is the data of two objects (c_1, c_2) in $\mathcal{C}$, and the product is denoted $c_1 \times c_2$. It is equipped with two projection morphisms $\pi_1 : c_1 \times c_2 \to c_1$ and $\pi_2 : c_1 \times c_2 \to c_2$, and for every object c' equipped with morphisms $\beta_1 : c' \to c_1$ and $\beta_2 : c' \to c_2$, there exists a unique morphism $\beta : c' \to c_1 \times c_2$ such that $\pi_1 \circ \beta = \beta_1$ and $\pi_2 \circ \beta = \beta_2$.

3.4.4

Let $\mathcal{I}$ be the category whose objects and non-identity morphisms are as depicted here:

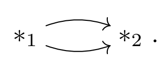


Thus a diagram $K : \mathcal{I} \to \mathcal{C}$ is the data of three objects (c_1, c_2, c) and two moprhisms $p_1 : c_1 \to c$ and $p_2 : c_2 \to c$. Notice that one can interpret a cone with vertex c' over such K as the data of morphisms $\pi_1 : c' \to c_1$ and $\pi_2 : c' \to c_2$ such that $p_1 \circ \pi_1 = p_2 \circ \pi_2$. The limit $\lim K$ is in this case called the **fiber product** of c_1 and c_2 over c, and is denoted $c_1 \underset{c}{\times} c_2$ (this notation is a bit ambigious, since p_1 and p_2 are not present in it; sometimes, if this is important, people write something like $c_1 \underset{p_1, c, p_2}{\times} c_2$ to clarify).

So, the fiber product is an object $c_1 \underset{c}{\times} c_2$ equipped with morphisms $\pi_1 : c_1 \underset{c}{\times} c_2 \to c_1$ and $\pi_2 : c_1 \underset{c}{\times} c_2 \to c_2$ such that $p_1 \circ \pi_1 = p_2 \circ \pi_2$. This satisfies the universal property, that given another object c' and morphisms $\pi'_1 : c' \to c_1$ and $\pi'_2 : c' \to c_2$ such that $p_1 \circ \pi'_1 = p_2 \circ \pi'_2$, there exists a unique morphism $\beta : c' \to c_1 \underset{c}{\times} c_2$ such that $\pi_1 \circ \beta = \pi'_1$ and $\pi_2 \circ \beta = \pi'_2$.

3.4.5

Let $\mathcal{I}$ be the category whose objects and non-identity morphisms are as depicted here:



Thus a diagram $K : \mathcal{I} \to \mathcal{C}$ is the data of two objects (c_1, c_2) and two morphisms $\alpha, \beta : c_1 \to c_2$. One can interpret a cone with vertex c' over K as the data of a morphism $\pi : c' \to c_1$ such that $\alpha \circ \pi = \beta \circ \pi$. The limit $\lim K$ is called the **equalizer** of α and β. It is an object $e \in \mathcal{C}$ equipped with a morphism $\pi : e \to c_1$ such that $\alpha \circ \pi = \beta \circ \pi$ with the following universal property: For every $c' \in \mathcal{C}$ equipped with a morphism $\pi' : c' \to c_1$ satisfying $\alpha \circ \pi' = \beta \circ \pi'$, there exists a unique morphism $\gamma : c' \to e$ such that $\pi' = \pi \circ \gamma$.

3.4.6

Let k be a field and consider the category Vec_k of vector spaces over k. Given a morphism $T : V \to W$ in Vec_k, the fiber product of

$$\begin{array}{ccc} & & V \\ & & \big\downarrow{\scriptstyle T} \\ 0 & \xrightarrow[0]{} & W \end{array}$$

and the equalizer of

$$V \underset{0}{\overset{T}{\rightrightarrows}} W$$

are the same. Indeed, both represent the functor sending a vector space U to the set of morphisms $S : U \to V$ satisfying $T \circ S = 0$. This limit is denoted $\mathrm{Ker}(T)$ (the **kernel** of T), and it comes equipped with a morphism $S : \mathrm{Ker}(T) \to V$. It satisfies the universal property, that $T \circ S = 0$, and given a morphism $S' : U \to V$ satisfying $T \circ S' = 0$ there exists a unique morphism $R : U \to \mathrm{Ker}(T)$ such that $S' = S \circ R$.

3.4.7

Let $\mathcal{J}$ be the poset of negative integers. Let $\mathbb{R}$ be the poset of real numbers. Then a diagram $\mathcal{J} \to \mathbb{R}$ is a decreasing sequence $r_{-1} \geq r_{-2} \geq \ldots$. The limit of this diagram exists if and only if the sequence is bounded below, and it is given by the infimum of this sequence, which is the same as its limit in the usual analysis sense.

3.4.8

Let us give a concrete example of a limit. Let p be a prime number. Consider the partially ordered set $\mathbb{Z}_{\geq 1}$, and view it as a category like we explained before. We have the functor $(\mathbb{Z}_{\geq 1})^{\mathrm{op}} \to \mathrm{CRng}$, sending n to $\mathbb{Z}/p^n\mathbb{Z}$, and $n \leq m$ is sent to $\mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ given by $x + p^m\mathbb{Z} \mapsto x + p^n\mathbb{Z}$. The limit of this functor is denoted by $\mathbb{Z}_p$, and is known as the **ring of p-adic numbers**. Thus, explicitly, an element of $\mathbb{Z}_p$ is a sequence $(x_n)_{n \in \mathbb{Z}_{\geq 1}}$ where $x_n \in \mathbb{Z}/p^n\mathbb{Z}$ (a residue of an integer modulo p^n), coordinated in the sense that given $n \leq m$, the residue of x_m modulo p^n is the same as x_n. Addition and multiplication are done component-wise. Such numbers arise when one wants to solve an equation such as $x^2 = 2$ in integers. There is no solution of course, but $x_1 := 3$ is a solution modulo 7: $x_1^2 \equiv_7 2$. One then can see that one can find an integer x_2 such that $x_2^2 \equiv_{7^2} 2$ and $x_2 \equiv_7 x_1$ (so x_2 recovers x_1 but is a solution of our equation to a greater precision), then an integer x_3 such that $x_3^2 \equiv_{7^3} 2$ and $x_3 \equiv_{7^2} x_2$, and so on. One obtains a "tower" which defined an "imaginary" integer, in the sense that we know which residues it leaves upon division by various powers 7^n (but there is no actual integer leaving all these residues). This is an element in the ring $\mathbb{Z}_7$. To conclude, the ideology has of course a strong commonality with that of representable functors, complex numbers etc. - instead of solving a problem, we create a bigger world with tailored solutions, and the object of study shifts to the study of the whole bigger world and its relations with the smaller world, and so on.

3.4.9

Let us remark here that $\mathrm{Cones}_K(c)$ can be seen as the limit of the functor $\mathcal{I} \to \mathrm{Set}$ given by $i \mapsto \mathrm{Hom}_{\mathcal{C}}(c, K(i))$. Then, with some circularity, one might rewrite the definition of a limit as having, for all $c \in \mathcal{C}$ and functorial in that c, bijections

$$\mathrm{Hom}_{\mathcal{C}}(c, \lim_{i \in \mathcal{I}} K(i)) \xrightarrow{\sim} \lim_{i \in \mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(c, K(i)).$$

In other words, we have some primordial notion of a limit in the category of sets, and then in a general category an object is a limit when we explicit how Hom into it is the limit of Hom's - Hom **into a limit is the limit of** Hom**'s**.

3.4.10

To exercise a bit, let us present the basics of the ideology of "working over a base", or the "relative point of view" (due to Grothendieck).

We fix a category $\mathcal{C}$, assume for simplicity that all finite products exist in $\mathcal{C}$. Of a morphism in $\pi_0 : d_0 \to c_0$ we think, informally, as a 'family of object in $\mathcal{C}$ parametrized by c_0, or living over c_0. Of c_0 we think as the "base" of the family and of d_0 as the "total space" of the family. If we have a "partial re-parametrization" $\alpha : c_1 \to c_0$, we form the fiber product

$$\begin{array}{ccc} d_1 & \longrightarrow & d_0 \\ \big\downarrow{\scriptstyle \pi_1} & & \big\downarrow{\scriptstyle \pi_0} \\ c_1 & \xrightarrow{\alpha} & c_0 \end{array}$$

and think of $\pi_1 : d_1 \to c_1$ as the original family, pull-backed to c_1 via α. One also calls the process of obtaining π_1 from π_0 "base change" (the base has changed, from c_0 to c_1).

Most clearly the intuition is seen when $\mathcal{C} = \mathrm{Set}$ (or $\mathcal{C} = \mathrm{Top}$). Then $\pi_0 : Y_0 \to X_0$ is informally the family of objects in $\mathcal{C}$ (i.e. sets) parametrized by points of X_0, where to each $x_0 \in X_0$ we associate the set $\pi_0^{-1}(x_0)$. Now if we have $\alpha : X_1 \to X_0$, we obtain a new family naturally, simply associating to $x_1 \in X_1$ the set $\pi_0^{-1}(\alpha(x_1))$. So we sample from the same family in a "re-parametrized" way (and also in a partial way potentially, if α is not surjective). Forming the fiber product

$$\begin{array}{ccc} Y_1 & \longrightarrow & Y_0 \\ \big\downarrow{\scriptstyle \pi_1} & & \big\downarrow{\scriptstyle \pi_0} \\ X_1 & \xrightarrow{\alpha} & X_0 \end{array} \quad ,$$

it is easy to see that we have a canonical bijection $\pi_0^{-1}(\alpha(x_1)) \cong \pi_1^{-1}(x_1)$, so indeed $\pi_1 : Y_1 \to X_1$ captures the new family.

When we have a property $\mathcal{P}$ of objects in $\mathcal{C}$, we would like to formulate a property $\mathcal{P}'$ of morphisms in the category $\mathcal{C}$ so that, informally, for a morphism has $\mathcal{P}'$ if and only if all its fibers have $\mathcal{P}$, in some "continuous"/"coherent" way. In particular, we always want the following two features:

- Denoting by $\bullet$ the final object in $\mathcal{C}$, we want $c \in \mathcal{C}$ to have $\mathcal{P}$ if and only if $c \to \bullet$ has $\mathcal{P}'$.
- ("stability under base change") Given a morphism $\pi : d_0 \to c_0$ which has $\mathcal{P}'$, and given any $\alpha : c_1 \to c_0$, forming the fiber product

$$\begin{array}{ccc} d_1 & \longrightarrow & d_0 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_0} \\ c_1 & \xrightarrow{\alpha} & c_0 \end{array} ,$$

  the morphism π_1 has $\mathcal{P}'$.

Oftentimes one also wants/has the following feature, basically saying that the total space of a $\mathcal{P}'$-family parametrized by a $\mathcal{P}$-base is $\mathcal{P}$:

- ("stability under composition") Given morphisms $\nu : e_0 \to d_0$ and $\pi : d_0 \to c_0$, if both ν and π have $\mathcal{P}'$ then $\pi \circ \nu$ has $\mathcal{P}'$.

As an example, we have the important property of a topological space, i.e. object in Top, to be compact. The relative property is defined as follows: A morphism $\pi : Y \to X$ in Top is **proper**, if for all compact $K \subset X$, the space $\pi^{-1}(K)$ is also compact. Then one checks that, indeed, the three properties above are satisfied. Notice that the fibers of a proper map are compact, but the inclusion of an open interval to a closed interval gives an example of a map all of whose fibers are compact, but which is not proper. This is why we said that having property $\mathcal{P}'$ should be thought of as all fibers having property $\mathcal{P}$ but in some "coherent" way, with no "abruptions".

3.5 Dualizing everything

By working with $\mathcal{C}^{\mathrm{op}}$ instead of $\mathcal{C}$, we obtain the category of pre-cosheaves $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$, the co-Yoneda lemma, corepresentability of functors, corepresenting objects, the functor of cocones $\mathrm{coCones}_K : \mathcal{C} \to \mathrm{Set}$, colimits. The colimit of $K : \mathcal{I} \to \mathcal{C}$ is denoted colim K or $\mathrm{colim}_{i \in \mathcal{I}} K(i)$. We have the **structural morphisms**, or **insertions/embeddings** $K(i) \to \mathrm{colim}\, K$ for every $i \in \mathcal{I}$. Thus, we have for the colimit bijections, functorial in $c \in \mathcal{C}$:

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}\,_{i \in \mathcal{I}} K(i), c) \xrightarrow{\sim} \mathrm{coCones}_K(c) \cong \lim_{i \in \mathcal{I}^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{C}}(K(i), c).$$

In words, a morphism from a colimit is a coordinated system of morhisms from the limitands, and the set of morphisms from a colimit is the limit of the sets of morphisms from the limitands (briefly - Hom from the colimit is the limit of Hom's).

3.6 Examples of colimits

3.6.1

For $\mathcal{I} = \emptyset$, one gets the notion of an **initial object** - an object such that, to any other object, there exists precisely one morphism from it.

3.6.2

If $\mathcal{I}$ is a set considered as a category with only identity morphisms, colim K is called the **coproduct**, denoted generally $\coprod_{i \in \mathcal{I}} K(i)$.

Let us consider $\mathcal{C} :=$ Set. Then $\coprod_{i \in \mathcal{I}} K(i)$ is the **disjoint union**. If we consider $\mathcal{C} := \mathrm{Mod}(R)$, the category of R-modules for a ring R (so for example vector spaces over a field $R := k$, or abelian groups (the case $R := \mathbb{Z}$)), then the coproduct is realized as the **direct sum** $\oplus_{i \in \mathcal{I}} K(i)$. It is the sub-module of the product $\prod_{i \in \mathcal{I}} K(i)$, consisting of vectors all of whose entries, except finitely many, are zero. The cocone morphisms $K(j) \to \oplus_{i \in \mathcal{I}} K(i)$ are given by sending $k \in K(j)$ to the vector whose j-th coordinate is k and all other coordinates are 0. If $\mathcal{C} :=$ Grp then again the description of coproducts is different. For example, the coproduct of $\mathbb{Z}$ and $\mathbb{Z}$ is a group, to give a morphism from which is the same as to give two elements. It is the free group on two generators, and we see that the construction of the coproduct must be a bit more sophisticated in this case.

So, we see that colimits tend to be "calculated differently" in the various standard categories (later we will be able to formalize this, by saying that the forgetful functors do not generally commute with colimits). This is a big difference between limits and colimits, at the base pointing to the asymmetry between subobjects and quotient objects in the axiomatics of sets.

3.6.3

Suppose that $\mathcal{C} :=$ Set and $\mathcal{I}$ is small, and let us give the concrete description of the colimit colim K. We first consider the disjoint union $C' := \coprod_{i \in \mathcal{I}} K(i)$ (let us write the element $k \in K(i)$ considered as an element of the disjoint union as (i, k)). We then consider the equivalence relation on this disjoint union generated by relations $(i_1, k) \sim (i_2, K(\alpha)(k))$ where $\alpha : i_1 \to i_2$ and $k \in K(i_1)$. We then consider the quotient C of the disjoint union by this equivalence relation. We have maps $K(i) \to C$ given by the obvious maps $K(i) \to C'$ (sending k to (i, k)) followed by the projection $C' \to C$. We leave as an exercise to check that this gives a universal cocone, i.e. makes C the colimit of K.

We can repeat a similar construction in some other categories, for example $\mathcal{C} := \mathrm{Top}$ (in particular, all small colimits exist in Top). Here, we will have to pay attention to topologies, recalling the disjoint union topology and then the quotient topology.

If we would like to describe a general small colimit in $\mathcal{C} := \mathrm{AbGrp}$, for example, then we proceed by first considering the direct sum $\oplus_{i \in \mathcal{I}} K(i)$ and then considering the quotient group of this by the subgroup generated by elements $(i_1, k) - (i_2, K(\alpha)(k))$ where $\alpha : i_1 \to i_2$ and $k \in K(i_1)$.

3.6.4

If $\mathcal{I}$ is the category whose objects and non-identity morphisms as follows:

$$\begin{array}{ccc} * & \longrightarrow & *_1 \, , \\ \downarrow & & \\ *_2 & & \end{array}$$

the colimit colim K is called the **push-out**, denoted $K(*_1) \coprod_{K(*)} K(*_2)$. Here one can illustrate graphically, for example how the push-out of two discs along their boundaries is a sphere.

3.6.5

Let $\mathcal{I}$ be the category whose objects and non-identity morphisms are as depicted here:

$$*_1 \rightrightarrows *_2 \, ,$$

the colimit colim K is called the **coequalizer**. Here one can illsutrate graphically, for example how the coequalizer of the two end points of an interval is the circle.

3.6.6

Again we can consider a morphism $T : V \to W$ in Vec_k, and the coequalizer of $T, 0 : V \to W$ is the same as the pushout $0 \coprod_V W$. This object is the **cokernel** of T, concretely given as the quotient of W by the image of T.

3.7 General colimits in terms of special ones

Let us choose to talk about colimits in this subsection, instead of limits. As before, there is no difference since we can always dualize, it is just that in the concrete examples colimits will be the more interesting choice to think about. Namely, we can motivate the discussion by the desire to show that in the category Grp there exist arbitrary small colimits (we have already seen that there exist arbitrary small limits in Grp by a concrete construction).

We say that a category is Hom-**small** if all Hom-sets in it are small.

Proposition 3.6. *Let $\mathcal{C}$ be a* Hom*-small category. Suppose that all small coproducts and all coequalizers exist in $\mathcal{C}$. Then all small colimits exist in $\mathcal{C}$.*

Proof. Let $\mathcal{I}$ be a small category and $K : \mathcal{I} \to \mathcal{C}$. We consider two morphisms

$$\alpha_0, \alpha_1 : \coprod_{i,j \in \mathcal{I},\ \beta : i \to j} K(i) \to \coprod_{i \in \mathcal{I}} K(i)$$

defined as follows. To define a morphism from the coproduct on the left, we need to define, for each $\beta : i \to j$, a morphism from $K(i)$. For α_0, we let it be the the structural inclusion of $K(i)$. For α_1, we let it be $K(\beta)$ followed by the structural inclusion of $K(j)$. We consider now the coequalizer E of α_0 and α_1. We have morphisms from $K(i)$'s to E, by the structural inclusion to the coproduct on the right followed by the structural morphism to the coequalizer. We leave to the reader to check that these maps form a cocone, and the universal property is satisfied. □

Thus, for example, to check that arbitrary small colimits in the category Grp exist,it is enough to check that arbitrary small coproducts exist, as well as coequalizers. It is easy to see what are coequalizers (quotient of the target by the normal subgroup generated by some quotients), so one is left with showing the existence of corpoducts.

A small category $\mathcal{I}$ is called **filtered**, if two conditions are satisfied. First, for every $i_1, i_2 \in \mathcal{I}$ there exists $i_3 \in \mathcal{I}$ and morphisms $i_1 \to i_3$ and $i_2 \to i_3$. Second, for every $i_1, i_2 \in \mathcal{I}$ and morphisms $\beta_1, \beta_2 : i_1 \to i_2$, there exists $i_3 \in \mathcal{I}$ and a morphism $\beta_3 : i_2 \to i_3$ such that $\beta_3 \circ \beta_1 = \beta_3 \circ \beta_2$. Small filtered colimits, that is colimits over a small filtered category, tend to be more straight-forward to calculate than general small colimits (intuitively speaking, the filtered colimit of objects is in some sense their "accumulation", rather than a more delicate "gluing"). For example, small filtered colimits in categories such as Grp and Mod(R) can be given the same description as those in Set (and in particular they exist). For example, let as consider Grp and let $K : \mathcal{I} \to \mathrm{Grp}$ be a small filtered diagram. We consider the set $\coprod_{i \in \mathcal{I}} K(i)$ (for clarity, let us write (i, k), where $i \in \mathcal{I}$ and $k \in K(i)$, for the element k in the disjoint union at the i-th place) and the equivalence relation on it - (i_1, k_1) and (i_2, k_2) are equivalent if there exist $\beta_1 : i_1 \to i_3$ and $\beta_2 : i_2 \to i_3$ such that $K(\beta_1)(k_1) = K(\beta_2)(k_2)$. One checks that this is indeed an equivalence relation. Then one defines a group structure on the set of equivalence classes as follows. Given (i_1, k_1) and (i_2, k_2), we find $i_3 \in \mathcal{I}$ and morphisms $\beta_1 : i_1 \to i_3$ and $\beta_2 : i_2 \to i_3$, and define the product of (i_1, k_1) and (i_2, k_2) to be $(i_3, K(\beta_1)(k_1) \cdot K(\beta_2)(k_2))$. One checks that this does not depend on the choice, etc. One then checks that the thus obtained group, together with the obvious morphisms from the various $K(i)$'s, gives the desired colimit.

Proposition 3.7. *Let $\mathcal{C}$ be a category. Suppose that all finite coproducts and all small filtered colimits exist in $\mathcal{C}$. Then all small coproducts exist in $\mathcal{C}$.*

Proof. Let I be a small set, and $(c_i)_{i\in I}$ a family of objects in $\mathcal{C}$. We want to show that the coproduct of this family exists. Let us consider the partially ordered set J whose elements are finite subsets of I and the partial order is that of containment. We consider the partially ordered set J as a category in our usual way. It is easy to see that J is a small filtered category. We have a functor $K : J \to \mathcal{C}$ defined as follows. We let $K(S)$ be the coproduct $\coprod_{i\in S} c_i$. We leave the reader to define precisely the functoriality of K. Let us remark momentarily that here we again face what we tried to stress - we "choose" a coproduct for every $S \in J$, a lot of choice in the standard axiomatics, but really no choice (a contractible category of choice). Now we claim that colim K, together with the morphisms from the c_i's given by $c_i \xrightarrow{\sim} K(\{i\}) \to \operatorname{colim} K$, is a coproduct, and leave it to the reader. □

We also have the following easier proposition:

Proposition 3.8. *Let $\mathcal{C}$ be a category. Suppose that it has an initial object and coproducts of any two objects. Then it has all finite coproducts.*

Proof. Left as an exercise. □

The last two propositions reduce further the statement that Grp contains all small colimits to showing that it has coproducts of two groups (as clearly there is an initial object - the trivial group). This coproduct can be described concretely, it is known in group theory as the "free product", denoted $G * H$. But later we will see also an abstract justification for the existence of $G * H$.

4 Adjoint functors

4.1 Bifunctors

When dealing with expressions like $\operatorname{Hom}_{\mathcal{C}}(c_1, c_2)$, we better get used to **bifunctors**.

Given categories $\mathcal{C}$ and $\mathcal{D}$, we can form the **product category** $\mathcal{C} \times \mathcal{D}$. Its objects are pairs (c, d) consisting of an object $c \in \mathcal{C}$ and an object $d \in \mathcal{D}$. Morphisms between (c_1, d_1) and (c_2, d_2) are pairs (α, β) consisting of a morphism $\alpha : c_1 \to c_2$ and a morphism $\beta : b_1 \to b_2$. Composition is again component-wise. We leave to the reader to see that the information of a functor

$$\mathbb{F} : \mathcal{D} \to \operatorname{Fun}(\mathcal{C}, \mathcal{E})$$

is "the same" as the information of a functor

$$\mathbb{F}^{\times} : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$$

where the relation is $\mathbb{F}^{\times}(c, d) = \mathbb{F}(d)(c)$ (so formally we have an identification of the categories $\operatorname{Fun}(\mathcal{D}, \operatorname{Fun}(\mathcal{C}, \mathcal{E}))$ and $\operatorname{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$; This identification is an equivalence of categories but in fact stronger (it is "propositional" in some sense)). A **bifunctor** is just the terminology for a functor out of a product of two categories.

Let us notice here that, given $G_1, G_2 : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$, a morphism $\alpha : G_1 \to G_2$ is a family $(\alpha_{c,d})_{c\in\mathcal{C}, d\in\mathcal{D}}$ of morphisms $\alpha_{c,d} : G_1(c,d) \to G_2(c,d)$ satisfying functoriality. We leave to the reader to see that functoriality in this case can be stated as the validity of two separate conditions, functoriality in c and functoriality in d. Functoriality in c means that given a morphism $\beta : c_1 \to c_2$ in $\mathcal{C}$ and $d \in \mathcal{D}$ the following square is commutative:

$$\begin{array}{ccc} G_1(c_1,d) & \xrightarrow{\alpha_{c_1,d}} & G_2(c_1,d) \\ {\scriptstyle G_1(\beta,\mathrm{id}_d)}\downarrow & & \downarrow{\scriptstyle G_2(\beta,\mathrm{id}_d)} \\ G_1(c_2,d) & \xrightarrow{\alpha_{c_2,d}} & G_2(c_2,d) \end{array},$$

and functoriality in d means that given a morphism $\beta : d_1 \to d_2$ in $\mathcal{D}$ and $c \in \mathcal{C}$ the following square is commutative:

$$\begin{array}{ccc} G_1(c,d_1) & \xrightarrow{\alpha_{c,d_1}} & G_2(c,d_1) \\ {\scriptstyle G_1(\mathrm{id}_c,\beta)}\downarrow & & \downarrow{\scriptstyle G_2(\mathrm{id}_c,\beta)} \\ G_1(c,d_2) & \xrightarrow{\alpha_{c,d_2}} & G_2(c,d_2) \end{array}.$$

4.2 The definition of adjoint functors

We have a functor

$$L_{\mathcal{C},\mathcal{D}} : \mathrm{Fun}(\mathcal{C},\mathcal{D})^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{D} \times \mathcal{C}^{\mathrm{op}}, \mathrm{Set})$$

given by sending $F : \mathcal{C} \to \mathcal{D}$ to $((d,c) \mapsto \mathrm{Hom}_{\mathcal{D}}(F(c),d))$. We leave to the reader to understand what we mean everything to do to morphisms, i.e. how we mean to complete the definition.

Similarly, we have a functor

$$R_{\mathcal{C},\mathcal{D}} : \mathrm{Fun}(\mathcal{D},\mathcal{C}) \to \mathrm{Fun}(\mathcal{D} \times \mathcal{C}^{\mathrm{op}}, \mathrm{Set})$$

given by sending $\mathcal{C} \leftarrow \mathcal{D} : G$ to $((d,c) \mapsto \mathrm{Hom}_{\mathcal{C}}(c, G(d)))$.

Given $F : \mathcal{C} \to \mathcal{D}$ and $\mathcal{C} \leftarrow \mathcal{D} : G$, an **adjunction** between F and G, making F the **left adjoint** of G and making G the **right adjoint** of F is, by definition, an isomorphism $\alpha : L_{\mathcal{C},\mathcal{D}}(F) \xrightarrow{\sim} R_{\mathcal{C},\mathcal{D}}(G)$. In other words, it is a family of bijections, for $c \in \mathcal{C}$ and $d \in \mathcal{D}$,

$$\alpha_{c,d} : \mathrm{Hom}_{\mathcal{D}}(F(c),d) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(c, G(d)),$$

functorial in c and d (as explicated above).

Let us discuss uniqueness of adjoints. Let us notice that the functors $L_{\mathcal{C},\mathcal{D}}$ and $R_{\mathcal{C},\mathcal{D}}$ are fully faithful. Indeed, for example discussing $R_{\mathcal{C},\mathcal{D}}$, notice that by identifying $\mathrm{Fun}(\mathcal{D} \times \mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ with $\mathrm{Fun}(\mathcal{D}, \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}))$, our functor $R_{\mathcal{C},\mathcal{D}}$ becomes the functor

$$(\mathrm{Ynd}_{\mathcal{C}} \circ -) : \mathrm{Fun}(\mathcal{D}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{D}, \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})).$$

By Lemma 2.12, this functor is fully faithful, and so $R_{\mathcal{C},\mathcal{D}}$ is fully faithful.

Since, looking at the defintion, a right adjoint of F can be thought of as an object in the fiber of $R_{\mathcal{C},\mathcal{D}}$ over $L_{\mathcal{C},\mathcal{D}}(F)$, and this fiber is either contractible or empty, we obtain the **uniqueness of right adjoints, up to a unique isomoprhism**: Given two right adjoints (G, α) and (G', α') of F, there exists a unique isomorphism $\beta : G \xrightarrow{\sim} G'$ such that

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(c, G(d)) & \xrightarrow{\alpha_{c,d}} & \\
{\scriptstyle \beta_d \circ -}\downarrow & & \mathrm{Hom}_{\mathcal{D}}(F(c), d) \\
\mathrm{Hom}_{\mathcal{C}}(c, G'(d)) & \xrightarrow{\alpha'_{c,d}} &
\end{array}$$

for all $c \in \mathcal{C}$ and $d \in \mathcal{D}$.

Symmetrically, we obtain the **uniqueness, up to a unique isomorphism, of left adjoints**.

Let us conduct a preliminary discussion of existence of adjoints, for example considering right adjoints. So let $F : \mathcal{C} \to \mathcal{D}$. Let us identify $\mathrm{Fun}(\mathcal{D} \times \mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ with $\mathrm{Fun}(\mathcal{D}, \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}))$ as above. As we mentioned, we then reinterpret $R_{\mathcal{C},\mathcal{D}}$ as composition with the Yoneda embedding

$$(\mathrm{Ynd}_{\mathcal{C}} \circ -) : \mathrm{Fun}(\mathcal{D}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{D}, \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}))$$

an the existence of a right adjoint for F means that $L_{\mathcal{C},\mathcal{D}}(F)$ sits in the essential image of this $(\mathrm{Ynd}_{\mathcal{C}} \circ -)$. By Lemma 2.12, $L_{\mathcal{C},\mathcal{D}}(F)$ sits in the essential image of $(\mathrm{Ynd}_{\mathcal{C}} \circ -)$ if and only if for any $d \in \mathcal{D}$, $L_{\mathcal{C},\mathcal{D}}(F)(d)$ sits in the essential image of $\mathrm{Ynd}_{\mathcal{C}}$, i.e. if and only if for any $d \in \mathcal{D}$ the functor

$$c \mapsto \mathrm{Hom}_{\mathcal{D}}(F(c), d)$$

is representable.

In other words, "object-wise existence of the adjoint implies its existence". This means that if we know that

$$c \mapsto \mathrm{Hom}_{\mathcal{D}}(F(c), d)$$

is representable for any $d \in \mathcal{D}$, we call a representing object by $G(d)$, and then we automatically obtain from these data what G should do to morphisms, so giving G the structure of a functor, which will have the structure of a right adjoint of F. It is good to understand this explicitly, but it is folded inside the above conceptualizations.

4.3 Some examples of adjoint functors

For a "forgetful" functor, the left adjoint is usually interpreted as a **"free construction"** (i.e. in the spirit of having generators without relations). For example, let k be a field. We have the forgetful functor $\underline{-} : \mathrm{Vec}_k \to \mathrm{Set}$. It has a left adjoint, which we have already described above: It is the functor $k[-] : \mathrm{Set} \to \mathrm{Vec}_k$ sending S to $k[S]$, a vector space with basis $(\delta_s)_{s \in S}$ (we have created a formal symbol δ_s for every $s \in S$). To construct the adjunction we need, given $S \in \mathrm{Set}$ and $V \in \mathrm{Vec}_k$, to construct a bijection

$$\alpha_{S,V} : \mathrm{Hom}_{\mathrm{Set}}(S, \underline{V}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Vec}_k}(k[S], V),$$

and then show that it is functorial in S and V. We construct $\alpha_{S,V}$ as sending $f : S \to \underline{V}$ to $T_f : k[S] \to V$ given by $T_f(\sum_{s \in S} c_s \cdot \delta_s) := \sum_{s \in S} c_s \cdot f(s)$. We leave to the reader to verify everything that is left. In words, this adjunction can be formulated as the usual linear algebra statement "to give a linear map from a vector space with a basis, is the same as to say where basis elements go".

The above example can be generalized as follows. Given a ring R, we consider the category $\mathrm{Mod}(R)$ of left R-modules. There is the forgetful functor $\underline{-} : \mathrm{Mod}(R) \to \mathrm{Set}$, and it has a left adjoint $R[-] : \mathrm{Set} \to \mathrm{Mod}(R)$ (everything is done the same as above). In particular, we can set $R := \mathbb{Z}$. Here $\mathrm{Mod}(\mathbb{Z})$ can be identified with the category of abelian groups. We obtain the construction of a free abelian group on a set of generators.

If we consider the forgetful functor $\underline{-} : \mathrm{Grp} \to \mathrm{Set}$ from the category of groups, it again has a left adjoint. It is the construction of a free group on a set of generators. This is more involved than the commutative examples above - the free group is constructed using words of arbitrary length etc. We will later see another approach, showing that the left adjoint exists abstractly, without a specific construction.

Let k be a field. We can consider the forgetful functor $\underline{-} : \mathrm{CAlg}_k \to \mathrm{Set}$ from commutative k-algebras. The left adjoint will send a set S to the k-algebra $P(S)$ of polynomials in variables $(x_s)_{s \in S}$. So, for example, for $S = \{1, \ldots, n\}$, we obtain the relation we have already mentioned above:

$$\underline{A}^n \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Set}}(\{1, \ldots, n\}, \underline{A}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{CAlg}_k}(k[x_1, \ldots, x_n], A).$$

Let k be a field. Consider this time the forgetul functor $\underline{-} : \mathrm{CAlg}_k \to \mathrm{Vec}_k$ from commutative k-algebras to k-vector spaces. It has a left adjoint, sending V to the "free symmetric algebra on V", whose invariant description requires to know a little bit about tensor products. However, we can proceed without, as follows. We want to describe the value of the left adjoint on a vector space V. Since every vector space admits a basis, we can replace V by an isomorphic vector space $k[S]$ for a set S. We then obtain:

$$\mathrm{Hom}_{\mathrm{Vec}_k}(k[S], A) \xleftarrow{\sim} \mathrm{Hom}_{\mathrm{Set}}(S, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{CAlg}_k}(P(S), A)$$

(we here follow the usual practice of omitting the name of the forgetful functor, it being implicit by observing in which category the Hom-set is taken), which shows that $k[S]$ is the value of the desired left adjoint on V. A further reflection on such a construction, involving a choice, is to think what happens when we choose another basis T for V. We will also get a realization of $P(T)$ as the value of the left adjoint on V. By our above discussions, we obtain a canonical isomorphism of k-algebras $P(S) \xrightarrow{\sim} P(T)$. One can trace the definitions and make the isomorphism explicit (we leave this to the reader).

The above examples where of functors **"forgetting extra structure"**. There are also functors **"forgetting extra property"**, whose left adjoints have the feeling of "projecting onto a full subcategory". As a first example, consider the category Grp of groups and its full subcategory AbGrp $\subset$ Grp of abelian groups. We denote by $I : \mathrm{AbGrp} \to \mathrm{Grp}$ this embedding. It has a left adjoint. To find it, one needs to think: Given a group G, what is an abelian group G' such that to give a homomorphism from G to any abelian group is the same as to give a homomorphism from G' to that abelian group? The answer is $G/[G,G]$, the quotient of G by the commutator subgroup (the subgroup generated by expressions $ghg^{-1}h^{-1}$). So, we leave the details: Construct a functor $G \mapsto G/[G,G]$ and show that it is left adjoint to I, i.e. construct functorial bijections

$$\mathrm{Hom}_{\mathrm{Grp}}(G, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{AbGrp}}(G/[G,G], A)$$

for a group G and an abelian group A.

Another example in the spirit of forgetting extra property is as follows. Consider the category $\mathrm{Met}_{...}$ of metric spaces, with morphisms being, say, maps $f : X \to Y$ for which there exists a constant $C > 0$ such that $d(f(x_1), f(x_2)) \leq C \cdot d(x_1, x_2)$ for all $x_1, x_2 \in X$. We have the full subcategory $I : \mathrm{Met}^{cmp}_{...} \subset \mathrm{Met}_{...}$ consisting of complete metric spaces. This inclusion I admits a left adjoint, the completion functor. It can be constructed by sending a metric space X to the the metric space of equivalence classes of Cauchy sequences in X. We leave the details.

One can also give plenty of negative examples. For example, The forgetful functor $G : \mathrm{Vec}_k \to \mathrm{Set}$ does not have a right adjoint. Indeed, we claim that $W := G^r(\{*_1, *_2\})$ (the value on a set with two elements) does not exist. This is an object equipped with bijections

$$\mathrm{Hom}_{\mathrm{Vec}_k}(V, W) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Set}}(G(V), \{*_1, *_2\}),$$

functorial in $V \in \mathrm{Vec}_k$. In words, W is a vector space, a linear map to which is the same as specifying a subset. Plugging in $V := 0$, we obtain a bijection between a set with one element and a set with two elements - a contradiction.

4.4 Adjoint functors in terms of units and counits

Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be adjoint functors, so we are given bijections

$$\alpha_{c,d} : \mathrm{Hom}_{\mathcal{D}}(F(c), d) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(c, G(d))$$

functorial in c and d. We then obtain a morphism

$$\epsilon_c := \alpha_{c,F(c)}(\mathrm{id}_{F(c)}) : c \to G(F(c)),$$

and we leave to the reader to check that it is functorial in c, so we obtain a morphism

$$\epsilon : \mathrm{Id}_{\mathcal{C}} \to G \circ F,$$

called the **unit** of the adjunction. Similarly, we have a morphism

$$\delta_d := \alpha^{-1}_{G(d),d}(\mathrm{id}_{G(d)}) : F(G(d)) \to d$$

which is functorial in d, so that we obtain a morphism

$$\delta : F \circ G \to \mathrm{Id}_{\mathcal{D}},$$

called the **counit** of the adjunction. We leave to the reader to verify that one has the following relation:

$$F \xrightarrow{F(\epsilon)} F \circ G \circ F \xrightarrow{\delta_F} F \quad \text{is equal to} \quad F \xrightarrow{\mathrm{id}_F} F$$

where the left expression uses a notation which is not hard to decipher, and explicitly it will yield the morphism that for $c \in \mathcal{C}$ is given by

$$F(c) \xrightarrow{F(\epsilon_c)} F(G(F(c))) \xrightarrow{\delta_{F(c)}} F(c).$$

Similarly, one also has

$$G \xrightarrow{\epsilon_G} G \circ F \circ G \xrightarrow{G(\delta)} G \quad \text{is equal to} \quad G \xrightarrow{\mathrm{id}_G} G.$$

One now also has the converse procedure. Namely, Suppose we are given functors $F : \mathcal{C} \to \mathcal{D}$ and $\mathcal{C} \leftarrow \mathcal{D} : G$ and morphisms

$$\epsilon : \mathrm{Id}_{\mathcal{C}} \to G \circ F, \quad \delta : F \circ G \to \mathrm{Id}_{\mathcal{D}}$$

satisfying

$$F \xrightarrow{F(\epsilon)} F \circ G \circ F \xrightarrow{\delta_F} F \quad \text{is equal to} \quad F \xrightarrow{\mathrm{id}_F} F$$

and

$$G \xrightarrow{\epsilon_G} G \circ F \circ G \xrightarrow{G(\delta)} G \quad \text{is equal to} \quad G \xrightarrow{\mathrm{id}_G} G.$$

Then we can form an adjunction between F and G, i.e. bijections

$$\alpha_{c,d} : \mathrm{Hom}_{\mathcal{D}}(F(c), d) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(c, G(d))$$

functorial in c and d, as follows. Given $\beta : F(c) \to d$, we need to produce $\gamma : c \to G(d)$. We do it by considering $c \xrightarrow{\epsilon_c} G(F(c)) \xrightarrow{G(\beta)} G(d)$. Conversely, given $\gamma : c \to G(d)$, we need to produce $\beta : F(c) \to d$. We do it by considering $F(c) \xrightarrow{F(\gamma)} F(G(d)) \xrightarrow{\delta_d} d$. We leave to the reader to check that those are mutually inverse.

One can now check that the two procedures, passing from α to (ϵ, δ) and vice versa, are mutually inverse.

4.5 Adjoint functors in terms of counits

There is also a third option for defining an adjunction between $F : \mathcal{C} \to \mathcal{D}$ and $\mathcal{C} \leftarrow \mathcal{D} : G$. Namely, one can define an adjunction as a morphism $\delta : F \circ G \to \mathrm{Id}_{\mathcal{D}}$ such that for all $c \in \mathcal{C}$ and $d \in \mathcal{D}$ the composition

$$\mathrm{Hom}_{\mathcal{C}}(c, G(d)) \to \mathrm{Hom}_{\mathcal{D}}(F(c), F(G(d)) \xrightarrow{\delta_d \circ -} \mathrm{Hom}_{\mathcal{D}}(F(c), d)$$

(where the first map is by applying F) is a bijection. We let the reader to complete the details of the equivalence of that definition with the previous ones (and, of course, one can also dualize and give a definition in terms of units).

4.6 Left adjoints to fully faithful functors

Here we present a piece of the formalization of examples above of left adjoints to forgetting extra property, such as the left adjoints to the embeddings of abelian groups in groups and complete metric spaces in metric spaces, in terms of counits.

Lemma 4.1. *Let $G : \mathcal{D} \to \mathcal{C}$ be a functor. Suppose that G admits a left adjoint F, with counit $\epsilon : F \circ G \to \mathrm{Id}_{\mathcal{D}}$. Then G is fully faithful if and only if ϵ is an isomorphism.*

Proof. Let $d_1, d_2 \in \mathcal{D}$. Let us consider the composition

$$\mathrm{Hom}_{\mathcal{D}}(d_1, d_2) \to \mathrm{Hom}_{\mathcal{D}}(F(G(d_1)), d_2) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(G(d_1), G(d_2))$$

where the first map is by precomposing with ϵ_{d_1} and the second map is by the adjunction of F and G. We now claim that the composition is equal to the map $\beta \mapsto G(\beta)$ given by the data of the functor G. This is an exercise in the definitions. Namely, denote by

$$\alpha_{c,d} : \mathrm{Hom}_{\mathcal{D}}(F(c), d) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(c, G(d))$$

the bijections of the adjunction. Let $\beta : d_1 \to d_2$. Then the image of β under the first of our maps is $\beta \circ \epsilon_{d_1} = \beta \circ \alpha^{-1}_{G(d_1),d_1}(\mathrm{id}_{G(d_1)})$ and then the image of that under the second of our maps is $\alpha_{G(d_1),d_2}(\beta \circ \alpha^{-1}_{G(d_1),d_1}(\mathrm{id}_{G(d_1)}))$, which is equal by the functoriality of α to $G(\beta) \circ \alpha_{G(d_1),d_1}(\alpha^{-1}_{G(d_1),d_1}(\mathrm{id}_{G(d_1)})) = G(\beta)$.

Now, from the above we see that G is fully faithful if and only if the first map above is a bijection for all d_1 and d_2. By Yoneda's lemma, this happens if and only if ϵ_{d_1} is an isomorphism for all d_1, i.e. if and only if ϵ is an isomorphism. $\square$

Exercise 4.1. *Formulate the lemma dual to Lemma 4.1 (with a right adjoint, the unit, etc.).*

Remark 4.2. The reader can return to examples of $\mathrm{AbGrp} \subset \mathrm{Grp}$ and $\mathrm{Met}_{\dots}^{cmp} \subset \mathrm{Met}_{\dots}$. The lemma says things like "the abelianization of an abelian group is canonically isomorphic to that group" and "the completion of a complete metric space is canonically isomorphic to that metric space".

Remark 4.3. The lemma is the analog of the statement that a map of sets is injective if and only if it admits a left inverse. However, in the categorical setting this left inverse has to be "regulated". So, as an exercise, you can find a functor which is not fully faithful and which admits a left inverse (a left inverse to a functor is a functor in the opposite direction and an isomorphism of the composition (in the relevant order) of these two with the identity functor). And also you can find a functor which is fully faithful but does not admit a left inverse.

Remark 4.4. Suppose that we have a full subcategory $\mathcal{C}^0 \subset \mathcal{C}$, and denote the embedding by I. Assume that I admits a left adjoint P. We can usually omit I, since $I(c)$ is the same as c (but thought of as objects of different categories), so the adjunction is given by isomorphisms $\mathrm{Hom}_{\mathcal{C}}(P(c_1), c_2) \cong \mathrm{Hom}_{\mathcal{C}}(c_1, c_2)$ for $c_1 \in \mathcal{C}$ and $c_2 \in \mathcal{C}^0$. Recall that given a finite-dimensional inner product space C and a subspace $C^0 \subset C$, we have a canonical projection linear operator $p : C \to C^0$, characterized by $\langle p(v), w \rangle = \langle v, w \rangle$ for $v \in C, w \in C^0$. This formula is similar to the adjunction formula above, and so it is not bad to think of P as a "projection" onto $\mathcal{C}^0$ (the most "efficient" one, as witnessed by the adjunction formula, with some similarity to the orthogonal projection being most "efficient" (for example in terms of minimizing some distances)). But, one should note that I might admit a right adjoint P', which will generally differ from P, and will be also a legitimate choice for a most "efficient" projection (so we have two choices and not one).

5 Limit and adjunction

5.1 Functors commuting with limits

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Let $K : \mathcal{I} \to \mathcal{C}$ be a diagram. Suppose that $\lim K$ exists. Recall that we have the canonical "universal" cone with vertex $\lim K$ over K, i.e. morphisms $\mathrm{pr}_i : \lim K \to K(i)$ for all $i \in \mathcal{I}$, coordinated in the suitable sense. Applying F, we clearly obtain a cone with vertex $F(\lim K)$ over $F \circ K$, by considering the morphisms $F(\mathrm{pr}_i) : F(\lim K) \to F(K(i))$ for all $i \in \mathcal{I}$. We can then ask whether this renders $F(\lim K)$ the limit of $F \circ K$, i.e. if this is a universal cone over $F \circ K$. If this is so, we say that F **commutes with the**

limit of K, or F **preserves the limit of** K. By the definitions, this means that for any $d \in \mathcal{D}$ the map

$$\operatorname{Hom}(d, F(\lim K)) \xrightarrow{\beta \mapsto (F(\mathrm{pr}_i) \circ \beta)_{i \in \mathcal{I}}} \mathrm{Cones}_{F \circ K}(d)$$

should be a bijection.

Remark 5.1. In other words, the cone with vertex $F(\lim K)$ over $F \circ K$ gives as a canonical morphism

$$F(\lim K) \to \lim(F \circ K)$$

(verbally if $\lim(F \circ K)$ exists, and working in $\operatorname{Fun}(\mathcal{D}^{\mathrm{op}}, \mathrm{Set})$ otherwise), and F commutes with the limit of K if this morphism is an isomorphism. To reiterate in a different graphical rendition, one asks whether the canonical morphism

$$F(\lim_{i \in \mathcal{I}} K(i)) \to \lim_{i \in \mathcal{I}} F(K(i))$$

is an isomorphism. Notice, in particular, that in our terminology, if F commutes with limit of K (so a-priori the limit of K is supposed to exist), then the limit of $F \circ K$ exists.

5.2 Right adjoints commute with limits

The following lemma is important:

Lemma 5.2. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be adjoint functors. Then G commutes with all limits which exist in $\mathcal{D}$, while F commutes with all colimits which exist in $\mathcal{C}$.*

Proof. It is enough to discuss G, as the discussion of F is symmetric. Let us denote by

$$\alpha_{c,d} : \operatorname{Hom}_{\mathcal{C}}(c, G(d)) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{D}}(F(c), d)$$

the adjunction data. We fix a diagram $K : \mathcal{I} \to \mathcal{D}$ and assume that its limit exists.

Given $c \in \mathcal{C}$ we would like to check whether the canonical map

$$\operatorname{Hom}(c, G(\lim_i K(i))) \xrightarrow{\beta \mapsto (G(\mathrm{pr}_i) \circ \beta)_{i \in \mathcal{I}}} \mathrm{Cones}_c(G \circ K)$$

is a bijection. Let us abbreviate $[-,-] := \operatorname{Hom}(-,-)$. Given $j \in \mathcal{I}$, we consider the following diagram of sets:

$$\begin{array}{ccccccc}
[c, G(\lim_i K(i))] & \xrightarrow{\alpha_{c, \lim K}} & [Fc, \lim_i K(i)] & \xrightarrow{\beta \mapsto (\mathrm{pr}_i \circ \beta)_{i \in \mathcal{I}}} & \mathrm{Cones}_K(Fc) & \longrightarrow & \mathrm{Cones}_{G \circ K}(c) \\
\downarrow{\scriptstyle G(\mathrm{pr}_j) \circ -} & & \searrow{\scriptstyle \mathrm{pr}_j \circ -} & & \swarrow{\scriptstyle \mathrm{pr}_j} & & \downarrow{\scriptstyle \mathrm{pr}_j} \\
[c, G(K(j))] & \xrightarrow{\alpha_{c, K(j)}} & & [Fc, K(j)] & & \xrightarrow{\alpha^{-1}_{c, K(j)}} & [c, G(K(j))]
\end{array}$$

The top right morphism in the diagram sends $(\beta_i : Fc \to K(i))_{i\in\mathcal{I}}$ to $(\alpha^{-1}_{c,K(i)}(\beta_i))_{i\in\mathcal{I}}$. The three maps on the top row are bijections, and thus the composition on the top row is a bijection. The three areas (two squares and one triangle) are commutative, one sees. Therefore considering the big outer square, we can write the following commutative triangle, whose top arrow is a bijection:

$$\begin{array}{ccc} [c, G(\lim_i K(i))] & \xrightarrow{\qquad\qquad} & \mathrm{Cones}_{G\circ K}(c)\ . \\ & {\scriptstyle G(\mathrm{pr}_j)\circ -}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_j} & \\ & [c, G(K(j))] & \end{array}$$

But this triangle precisely means that the top map is the one we wanted to check is an bijection, so we are done. $\square$

Remark 5.3. Recall that we can interpret $\mathrm{Cones}_K(c)$ as $\lim_{i\in\mathcal{I}} \mathrm{Hom}(c, K(i))$. Then someone used to categories might want to abbreviate the last proof to:

$$\mathrm{Hom}_{\mathcal{C}}(c, G(\lim_{i\in I} K(i))) \cong \mathrm{Hom}_{\mathcal{D}}(F(c), \lim_{i\in I} K(i)) \cong \lim_{i\in I} \mathrm{Hom}_{\mathcal{D}}(F(c), K(i)) \cong$$

$$\cong \lim_{i\in I} \mathrm{Hom}_{\mathcal{C}}(c, G(K(i)) \cong \mathrm{Hom}_{\mathcal{C}}(c, \lim_{i\in I} G(K(i))),$$

deducing from Yoneda's lemma that $G(\lim_{i\in I} K(i)) \cong \lim_{i\in I} G(K(i))$. Of course, one should be careful when making such an abbreviation, for various reasons.

5.3 Cofinality and limits

Let $L : \mathcal{J} \to \mathcal{I}$ be a functor. Given a diagram $K : \mathcal{I} \to \mathcal{C}$ and an object $c \in \mathcal{C}$, every cone with vertex c over K defines in a clear way a cone with vertex c over $K \circ L$. This is functorial in c. So we obtain a morphism

$$\mathrm{Cones}_K \to \mathrm{Cones}_{K\circ L},$$

and if those are representable a morphism

$$\lim K \to \lim(K \circ L).$$

We want conditions on L such that for all K this will be an isomorphism.

Let us say that L is **initial**, or **cofinal** if the following two conditions are satisfied:

- For every $i \in \mathcal{I}$ there exists a $j \in \mathcal{J}$ and a morphism $\gamma : L(j) \to i$.
- For $i \in \mathcal{I}$, $j, j' \in \mathcal{J}$ and morphisms $\gamma : L(j) \to i$ and $\gamma' : L(j') \to i$, there exists $j'' \in \mathcal{J}$ and morphisms $\delta : j'' \to j$ and $\delta' : j'' \to j'$ such that $\gamma' \circ L(\delta') = \gamma \circ L(\delta)$.

Lemma 5.4. *Suppose that $L : \mathcal{J} \to \mathcal{I}$ is initial. Let $K : \mathcal{I} \to \mathcal{C}$ be a diagram. Then the morphism above*

$$\mathrm{Cones}_K \to \mathrm{Cones}_{K \circ L}$$

is an isomorphism.

Proof. Let us be given a cone $(\beta_j : c \to K(L(j)))_{j \in \mathcal{J}}$ with vertex c over $K \circ L$. We want to define a cone $(\alpha_i : c \to K(i))_{i \in \mathcal{I}}$ with vertex c over K. Since L is initial, given $i \in \mathcal{I}$ there exists $j \in \mathcal{J}$ and a morphism $\gamma : L(j) \to i$. Let us define $\alpha_i := K(\gamma) \circ \beta_j$. One needs to check that this is well-defined. Namely, suppose that we have $j' \in \mathcal{J}$ and a morphism $\gamma' : L(j') \to i$. We need to check that $K(\gamma') \circ \beta_{j'} = K(\gamma) \circ \beta_j$. Since L is initial, there exists $j'' \in \mathcal{J}$ and morphisms $\delta : j'' \to j$ and $\delta' : j'' \to j'$ such that $\gamma' \circ L(\delta') = \gamma \circ L(\delta)$. Thus

$$K(\gamma') \circ \beta_{j'} = K(\gamma') \circ K(L(\delta')) \circ \beta_{j''} = K(\gamma' \circ L(\delta')) \circ \beta_{j''} =$$

$$= K(\gamma \circ L(\delta)) \circ \beta_{j''} = K(\gamma) \circ K(L(\delta)) \circ \beta_{j''} = K(\gamma) \circ \beta_j.$$

Having defined the map between the sets of cones in the opposite direction, it is now easy, in the same spirit, to see that it is inverse to our map. □

Remark 5.5. Suppose that $L : \mathcal{J} \to \mathcal{I}$ is initial. Let $K : \mathcal{I} \to \mathcal{C}$ be a diagram. Suppose that $\lim K$ exists (and so by the above $\lim(K \circ L)$ exists as well). Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We then notice that F commutes with limit of K if and only if F commutes with the limit of $K \circ L$. Indeed, we have the following commutative diagram of objects in $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathrm{Set})$:

$$\begin{array}{ccc}
\mathrm{Ynd}_{\mathcal{D}}(F(\lim K)) & \longrightarrow & \mathrm{Cones}_{F \circ K} \\
\downarrow & & \downarrow \\
\mathrm{Ynd}_{\mathcal{D}}(F(\lim(K \circ L))) & \longrightarrow & \mathrm{Cones}_{F \circ K \circ L}
\end{array}$$

where the horizontal morphisms are the comparison morphisms we discussed when defining the commutation of functors with limits and the vertical morphisms are as we currently discuss. As we showed, since L is initial both vertical morphisms are isomorphisms. Hence the upper horizontal morphism is an isomorphism if and only if the lower horizontal morphism is.

5.4 A detour on retracts and idempotents

Let $\mathcal{C}$ be a category. Let $c \in \mathcal{C}$. By a **retract of** c we will mean a triple (e, i, r) consisting of $e \in \mathcal{C}$ and morphisms $i : e \to c$ and $r : c \to e$ such that $r \circ i = \mathrm{id}_e$. One naturally defines the category of retracts of c (morphisms are morphisms between the e's commuting with the i's and r's). A morphism $\alpha : c \to c$ is called an **idempotent** (or **projector**) if $\alpha \circ \alpha = \alpha$. Given a retract (e, i, r) of c, we have an idempotent $i \circ r : c \to c$. We call a retract (e, i, r) an **image** of the idempotent α if $\alpha = i \circ r$.

Lemma 5.6. *Let $\mathcal{C}$ be a category, $c \in \mathcal{C}$ and $\alpha : c \to c$ an idempotent. Then the full subcategory of the category of retracts of c, consisting of images of α, is contractible.*

Proof. Given two images (e, i, r) and (e', i', r') of α, a morphism $\beta : e \to e'$ between these images will have to satisfy $i' \circ \beta = i$ and therefore $\beta = r' \circ i' \circ \beta = r' \circ i$. Conversely, $\beta := r' \circ i$ provides a morphism: $i' \circ (r' \circ i) = i$ and $(r' \circ i) \circ r = r' \circ \alpha = r' \circ (i' \circ r') = r'$. □

Lemma 5.7. *Let $\mathcal{C}$ be a category, $c \in \mathcal{C}$ and $\alpha : c \to c$ an idempotent.*

1. *Suppose that (e, i, r) is a retract of c which is an image of α. Then $i : e \to r$ is an equalizer of* $c \underset{\mathrm{id}}{\overset{\alpha}{\rightrightarrows}} c$.
2. *Suppose that there exists an equalizer $i : e \to c$ of* $c \underset{\mathrm{id}}{\overset{\alpha}{\rightrightarrows}} c$ *. Then there exists a unique $r : c \to e$ such that (e, i, r) is a retract of c which is an image of α.*

Proof. (1) Clearly $\alpha \circ i = i$. Let $f : x \to c$ be such that $\alpha \circ f = f$. We want to show that there exists a unique $f' : x \to e$ such that $i \circ f' = f$. Composing with r, we see that $f' = r \circ f$. Conversely, we indeed have $i \circ (r \circ f) = \alpha \circ f = f$.

(2) We have the map $\alpha : c \to c$ which satisfies $\alpha \circ \alpha = \alpha$ and therefore by the universal property of the equalizer there exists a unique $r : c \to e$ such that $i \circ r = \alpha$. We want to also check that $r \circ i = \mathrm{id}_e$. Notice that i is a monomorphism (this in general holds for an equalizer, as the universal property shows that composing with i is injective on Hom-sets). Therefore, to check that $r \circ i = \mathrm{id}_e$, it is enough to check that $i \circ (r \circ i) = i \circ \mathrm{id}_e$. The right hand side is i, while the left hand side is $i \circ (r \circ i) = (i \circ r) \circ i = \alpha \circ i = \mathrm{id}_c \circ i = i$ as well. □

Remark 5.8. Notions of retracts, idempotents, images of idempotents are "better" than limits in the following sense. For a functor to commute with limits is a request which is not automatic. However, let $F : \mathcal{C} \to \mathcal{D}$ be a functor and let $c \in \mathcal{C}$. If (e, i, r) is a retract of c, then it is immediate to see that $(F(e), F(i), F(r))$ is a retract of $F(c)$. If $\alpha : c \to c$ is an idempotent, then $F(\alpha) : F(c) \to F(c)$ is an idempotent. If (e, i, r) is a retract of c which is the image of an idempotent $\alpha : c \to c$, then $(F(e), F(i), F(r))$ is a retract of $F(c)$ which is the image of the idempotent $F(\alpha) : F(c) \to F(c)$. In addition, in view of the lemma above, it is also therefore easy to see that F commutes with equalizers of $c \underset{\mathrm{id}}{\overset{\alpha}{\rightrightarrows}} c$ whenever $\alpha : c \to c$ is an idempotent. So all functors commute with some very specific limits (and dually, colimits). Fundamentally, this is because of the basic distinction, that in the current case one can write everything in terms of morphisms and some equations that their compositions satisfy - all functors preserve such relations.

Lemma 5.9. *Let $\mathcal{C}$ be a category, and suppose that images of idempotents exist in $\mathcal{C}$. Let $F \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ and assume that F is the retract of a representable functor. Then F is representable.*

Proof. Let $c \in \mathcal{C}$ be such that F is a retract of $\mathrm{Ynd}_{\mathcal{C}}(c)$. Form the idempotent $\alpha : \mathrm{Ynd}_{\mathcal{C}}(c) \to \mathrm{Ynd}_{\mathcal{C}}(c)$ whose image F is. By Yoneda's lemma, there exists a unique $\beta : c \to c$ such that $\mathrm{Ynd}_{\mathcal{C}}(\beta) = \alpha$, and clearly β is an idempotent. Consider the image e of β, which exists by assumption. Then, as we remarked, $\mathrm{Ynd}_{\mathcal{C}}(e)$ is the image of $\mathrm{Ynd}_{\mathcal{C}}(\beta) = \alpha$, so by a lemma above, we see that $\mathrm{Ynd}_{\mathcal{C}}(e)$ is isomorphic to F. ☐

5.5 The general AFT (adjoint functor theorem)

Let $\mathcal{C} \leftarrow \mathcal{D} : G$ be a functor. We want to find some general conditions for the existence of its left adjoint.

First we introduce some notation. Given $c \in \mathcal{C}$, let us consider the category $G^{\leftarrow}(c)$ whose objects are pairs (d, α) consisting of $d \in \mathcal{D}$ and $\alpha : c \to G(d)$, and morphisms between (d, α) and (d', α') are morphisms $\beta : d \to d'$ such that $G(\beta) \circ \alpha = \alpha'$. We have the "forgetful" functor $\mathrm{Frg}_{\mathcal{D}}^{G^{\leftarrow}(c)} : G^{\leftarrow}(c) \to \mathcal{D}$ given by $(d, \alpha) \mapsto d$. Let us abbreviate $K := \mathrm{Frg}_{\mathcal{D}}^{G^{\leftarrow}(c)}$ in this subsection. (remember to rewrite notations)

Lemma 5.10. *Let $c \in \mathcal{C}$. Let us suppose that $\lim K$ exists, and that G commutes with this limit. Suppose also that images of idempotents exist in $\mathcal{D}$. Then $G^l(c)$ exists (it is the image of some idempotent on $\lim K$).*

Proof. Notice that we have a tautological morphism $c \to \lim(G \circ K)$ - to define such a morphism we need for every $(d, \alpha) \in G^{\leftarrow}(d)$ to define a morphism $c \to G(K((d, \alpha)) = G(d)$ so that the result family of morphisms is compatible; α gives such a morphism and it is straight-forward to check the compatibility. Therefore, by our assumption that $G(\lim K) \xrightarrow{\sim} \lim(G \circ K)$ we obtain a morphism $\gamma : c \to G(\lim K)$. We now consider, for $d \in \mathcal{D}$, the map

$$\delta_d : \mathrm{Hom}_{\mathcal{D}}(\lim K, d) \to \mathrm{Hom}_{\mathcal{C}}(c, G(d))$$

given by sending $\beta : \lim K \to d$ to $G(\beta) \circ \gamma$. Clearly this is functorial in d. On other hand, we also have a map

$$\mathrm{Hom}_{\mathcal{D}}(\lim K, d) \leftarrow \mathrm{Hom}_{\mathcal{C}}(c, G(d)) : \epsilon_d$$

given by, given a morphism $\alpha : c \to G(d)$, constructing the object $(d, \alpha) \in G^{\leftarrow}(c)$ and then the morphism $\mathrm{pr}_{(d,\alpha)} : \lim K \to K((d, \alpha)) = d$. This is also functorial in d. We obtain morphisms in $\mathrm{Fun}(\mathcal{D}, \mathrm{Set})$:

$$\delta : \mathrm{Hom}_{\mathcal{D}}(\lim K, -) \rightleftarrows \mathrm{Hom}_{\mathcal{C}}(c, G(-)) : \epsilon,$$

and we leave to the reader to check that one has $\delta \circ \epsilon = \mathrm{id}$. This means that the functor $\mathrm{Hom}_{\mathcal{C}}(c, G(-))$ is a retract of the representable functor $\mathrm{Hom}_{\mathcal{D}}(\lim K, -)$.

By Lemma 5.9, since we assume that images of idempotents exist in $\mathcal{D}$, we deduce that $\mathrm{Hom}_{\mathcal{C}}(c, G(-))$ is representable, which by definition is the existence of $G^l(c)$. ☐

The problem with this "abstract non-sense" lemma is that even if dealing with categories such as Set, where all limits of small diagrams exist, it is not clear whether a limit such as $\lim K$ above exists, since the category $G^{\leftarrow}(c)$ is not small (one can not form the product of small sets parametrized by a class and still have a small set). Even if in some approach one would say that one is willing to have "bigger" sets as part of Set to ensure the existence of more limits (as is basically done in the conception of **Grothendieck's universes**), we will have the problem of a snake eating its tail, as the category $G^{\leftarrow}(c)$ becomes accordingly bigger. The solution is to be able to reduce the size of the category $G^{\leftarrow}(c)$ while keeping the same limit. At the base of this is the important philosophy that **the category is big (and has some ugly objects), but it is determined by a small subcategory (of nice objects)**.

Let us say that G **satisfies the solution set condition** if for every $c \in \mathcal{C}$ there exists a small full subcategory $\mathcal{D}^0 \subset \mathcal{D}$ such that for every $d \in \mathcal{D}$ and a morphism $\alpha : c \to G(d)$ we can find $d_0 \in \mathcal{D}^0$, a morphism $\alpha' : c \to G(d_0)$ and a morphism $\beta : d_0 \to d$ such that $G(\beta) \circ \alpha' = \alpha$.

Theorem 5.11 (General adjoint functor thoerem)**.** *Let $G : \mathcal{D} \to \mathcal{C}$ be a functor. Suppose that all* Hom*-sets in $\mathcal{D}$ are small, and that all small limits exist in $\mathcal{D}$. Then G admits a left adjoint if and only if G commutes with all small limits and G satisfies the solution set condition.*

Proof. Assume that G admits a left adjoint F. We already saw that G then commutes with all small limits. Let us show that G satisfies the solution set condition. Let $c \in \mathcal{C}$. We take $\mathcal{D}^0$ to be the full subcategory of $\mathcal{D}$ spanned by the one object $F(c)$. For $d \in \mathcal{D}$ and $\alpha : c \to G(d)$, the adjunction corresponds to α a morphism $\beta : F(c) \to d$, and we leave as an exercise in manipulating with the data of an adjunction that α is equal to the composition of the unit morphism $c \to G(F(c))$ with $G(F(c)) \xrightarrow{G(\beta)} d$.

Now we get to the main part, showing the converse. Let us fix $c \in \mathcal{C}$. We want to establish the existence of $G^l(c)$ using Lemma 5.10. Image of idempotents exist in $\mathcal{C}$ as all small limits exist in $\mathcal{C}$. If we find an initial functor $L : \mathcal{I} \to G^{\leftarrow}(c)$ with $\mathcal{I}$ small, then by assumption $\lim(K \circ L)$ exists, and therefore $\lim K$ exists. Moreover, by assumption G commutes with the limit of $K \circ L$, and therefore G commutes with the limit of K. So it is enough to show the existence of such L. Let $\mathcal{D}^0 \subset \mathcal{D}$ be as in the definition of the solution set condition and denote by $I : \mathcal{D}^0 \to \mathcal{D}$ the inclusion functor. The category $(G \circ I)^{\leftarrow}(c)$ is obviously small, and we have the obvious inclusion functor $L : (G \circ I)^{\leftarrow}(c) \to G^{\leftarrow}(c)$. It is left to check that L is initial. The first condition in the definition of an initial functor is, when unfoldedin our case, precisely the solution set condition. As for the second condition, we are given $c \xrightarrow{\alpha} G(d)$, $c \xrightarrow{\alpha_1} G(d_1)$, $c \xrightarrow{\alpha_2} G(d_2)$ and $\beta_1 : d_1 \to d$, $\beta_2 : d_2 \to d$ such that $G(\beta_1) \circ \alpha_1 = \alpha$ and $G(\beta_2) \circ \alpha_2 = \alpha$ (here $d_1, d_2 \in \mathcal{D}^0$).

We want to see that there exists $c \xrightarrow{\alpha_3} G(d_3)$ (where $d_3 \in \mathcal{D}^0$) and morphisms $d_3 \xrightarrow{\gamma_1} d_1$ and $d_3 \xrightarrow{\gamma_2} d_2$ such that $G(\gamma_1) \circ \alpha_3 = \alpha_1$ and $G(\gamma_2) \circ \alpha_3 = \alpha_2$ and $\beta_1 \circ \gamma_1 = \beta_2 \circ \gamma_2$. Let us first consider the object $d_{12} := d_1 \underset{d}{\times} d_2 \in \mathcal{D}$ (where the structural morphisms of the fiber product are β_1 and β_2), with its two projections $p_1 : d_{12} \to d_1$ and $p_2 : d_{12} \to d_2$. Since G commutes with small limits, and in particular with fiber products, we have a unique morphism $\alpha_{12} : c \to G(d_{12})$ such that $G(p_1) \circ \alpha_{12} = \alpha_1$ and $G(p_2) \circ \alpha_{12} = \alpha_2$. Now, again by the solution set condition, we can find some $c \xrightarrow{\alpha_3} G(d_3)$ (with $d_3 \in \mathcal{D}^0$) and a morphism $\epsilon : d_3 \to d_{12}$ such that $G(\epsilon) \circ \alpha_3 = \alpha_{12}$. Then, in fact, by setting $\gamma_1 := p_1 \circ \epsilon$ and $\gamma_2 := p_2 \circ \epsilon$ we obtain what we wanted.

□

5.6 Free groups

Let us consider the forgetful functor Set ← Grp : Frg. All conditions of the general adjoint functor theorem are clearly satisfied, except the solution set condition. For that, let $S \in$ Set. What we need to check is the existence of a small full subcategory $\mathcal{K} \subset$ Grp such that for any group G and a morphism of sets $f : S \to G$, there exists $G' \in \mathcal{K}$, a morphism of groups $\theta : G' \to G$ and a morphism of sets $f' : S \to G'$ such that $f = \theta \circ f'$. Let κ be the cardinality of S if S is infinite, and $\aleph_0$ if S is finite. We can fix a set T of cardinality κ and consider $\mathcal{K}$ to be the set of groups with underlying set being a subset of T. Given our $f : S \to G$, the subgroup $H \subset G$ generated by $f(S)$ has cardinality $\leq \kappa$. Therefore this subgroup is isomorphic to some $G' \in \mathcal{K}$. Thus, fixing such an isomorphism $\epsilon : H \xrightarrow{\sim} G'$, our $f : S \to G$ factors as $\theta \circ f'$ with $f' : S \xrightarrow{f} f(S) \subset H \xrightarrow{\epsilon} G'$ and $\theta : G' \xrightarrow{\epsilon^{-1}} H \subset G$.

Thus, the forgetful functor Set ← Grp : Frg admits a left adjoint Fre : Set → Grp. The group Fre(S) is the **free group on** S. It admits a map of sets $i : S \to \text{Fre}(S)$ (this is the unit of the adjunction), and for every group G together with a map of sets $f : S \to G$, there exists a unique group homomorphism $\theta : \text{Fre}(S) \to G$ such that $f = \theta \circ i$. One can see that i injective, for example as follows. Given $s \in S$, we can consider some non-trivial group G and consider the function $f : S \to G$ sending s to some $1 \neq g \in G$ and all the other elements of S to 1. Then writing $f = \theta \circ i$ as above, we clearly obtain that s is mapped under i to a different element than the element to which any other element of S is mapped.

There is also an explicit construction of Fre(G), using formal words in elements of S and their formal inverses. One can ponder regarding the two approaches, how the general adjoint functor theorem were able to provide an object without constructing it, and so on.

Let us now see, as promised somewhere above, that given groups G_1, G_2, that their coproduct, denoted $G_1 * G_2$ and called the free product in this context (as already mentioned above), exists. Recall that above we saw that this is all what is needed in order to deduce the existence of arbitrary small colimits in the category Grp. First, suppose that $\alpha : H_1 \to G_1$ is a surjective homomorphism and that $H_1 * G_2$ was shown to exist (and we will show that $G_1 * G_2$ then exists). Denote by $K \subset H_1$ the kernel of α. Denote by $i : H_1 \to H_1 * G_2$ and $j : G_2 \to H_1 * G_2$ the structural insertions. Denote by $N \subset H_1 * G_2$ the normal subgroup generated by $i(K)$. We have $(H_1 * G_2)/N$ equipped with the homomorphisms $i' : G_1 \cong H_1/K \to (H_1 * G_2)/N$ and $j' : G_2 \to (H_1 * G_2)/N$ clearly obtained using i and j, and we claim that this furnishes the coproduct of G_1 and G_2. Indeed, to give a homomorphism from $(H_1 * G_2)/N$ is the same as to give a homomorphism from $H_1 * G_2$ which is trivial on N, which is the same as to give a homomorphism from $H_1 * G_2$ which is trivial on $i(K)$, which is the same as to give a homomorphism from $H_1 * G_2$ which when precomposed with i is trivial on K. Since $H_1 * G_2$ is a coproduct, this is the same as giving homomorphisms from H_1 and G_2, the first one being trivial on K, and this is the same as giving homomorphisms from G_1 and G_2 (let us leave the reader to formalize this). Of course, we have the symmetric thing for G_2, so when showing the existence of the coproduct we can replace G_1 and G_2 by groups surjecting homomorphically to them. Let $S_1 \subset G_1$ and $S_2 \subset G_2$ be generating subsets (for example, we can take $S_i := G_i$). By adjunction, corresponding to the inclusion maps of sets $S_1 \to G_1$ and $S_2 \to G_2$, we obtain homomorphisms of groups $\mathrm{Fre}(S_1) \to G_1$ and $\mathrm{Fre}(S_2) \to G_2$. Those are surjective (does the reader see why?). Therefore, as we just explained, it is enough to see that $\mathrm{Fre}(S_1) * \mathrm{Fre}(S_2)$ exists. However, Fre, being a left adjoint, preserves all small colimits (which exist in its domain). Therefore $\mathrm{Fre}(S_1 \coprod S_2)$ is in fact a realization of $\mathrm{Fre}(S_1) * \mathrm{Fre}(S_2)$.

5.7 Stone-Cech compactification

Let us consider the forgetful functor $\mathrm{Top} \leftarrow \mathrm{Top}^{ch} : \mathrm{Frg}$, where Top^{ch} is the full subcategory of Top consisting of compact Hausdorff topological spaces. Again one verifies the conditions of the general adjoint functor theorem, very similarly to the above example with its cardinality trick, replacing the smallest normal subgroup construction by the closure construction (let us note, and leave as an exercise for the reader, that small limits in Top^{ch} exist and are the same as in Top, and so are respected by the forgetful functor - one needs to use Tychonoff's theorem, and seeing that the limit is realized as a closed subspace of a product).

Therefore there exists a left adjoint of Frg, a functor $SC : \mathrm{Top} \to \mathrm{Top}^{ch}$, called the **Stone-Cech compactification**. So, $SC(X)$ is in some sense the "best" compact Hausdorff space with a map from X, and we have bijections

$$\mathrm{Hom}_{\mathrm{Top}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Top}^{ch}}(SC(X), Y)$$

functorial in $Y \in \mathrm{Top}^{ch}$.

Again we note that there are several explicit constructions of $SC(S)$ (which we currently omit), and it is interesting to contrast the topical intuition/"expertise" needed to perform/figure out the explicit constructions, with how we simply characterized uniquely this space and showed its existence, while preserving our ignorance.

As a corollary, one can deduce the existence of arbitrary small colimits in Top^{ch} (provided that we performed the easier exercise of establishing the existence of arbitrary small colimits in Top). Indeed, let $K : \mathcal{I} \to \mathrm{Top}^{ch}$ be a small diagram. We construct the following bijecions, functorial in $Y \in \mathrm{Top}^{ch}$:

$$\mathrm{Hom}_{\mathrm{Top}^{ch}}(SC(\mathrm{colim}\ _{i\in\mathcal{I}}\ \mathrm{Frg}(K(i))), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Top}}(\mathrm{colim}\ _{i\in\mathcal{I}}\ \mathrm{Frg}(K(i)), \mathrm{Frg}(Y)) \xrightarrow{\sim}$$

$$\xrightarrow{\sim} \lim_{i\in\mathcal{I}} \mathrm{Hom}_{\mathrm{Top}}(\mathrm{Frg}(K(i)), \mathrm{Frg}(Y)) \xrightarrow{\sim} \lim_{i\in\mathcal{I}} \mathrm{Hom}_{\mathrm{Top}^{ch}}(K(i), Y)$$

(the first bijection is by adjunction of SC and Frg, the second bijection is by the structure of a colimit, the third bijection is by Frg being, by definition, fully faithful). The composed bijection gives $SC(\mathrm{colim}\ _{i\in\mathcal{I}}\ \mathrm{Frg}(K(i)))$ the structure of the colimit of K in Top^{ch}. Thus, less formally, to compute the colimit of compact Hausdorff spaces X_i in Top^{ch}, we compute their colimit in Top and apply SC.